November 19, 2024

Problem Set 9

Exercise 1. Let V be a complex vector space of dimension k and consider $V^{\otimes n}$. Then $V^{\otimes n}$ is a representation of the symmetric group defined as

$$\phi(\sigma)\,(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = u_{\sigma(1)} \otimes u_{\sigma(2)} \ldots \otimes u_{\sigma(n)}.$$

By Maschke's theorem, $V^{\otimes n}$ decomposes as a direct sum of S_n- irreducible representations. The following exercise identifies the subspaces of $V^{\otimes n}$ that carry the trivial (resp. sign) isotypical component with respect to the S_n action.

(a) Define $P_+ : V^{\otimes n} \to V^{\otimes n}$ by

$$P_+(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes u_{\sigma(2)} \ldots \otimes u_{\sigma(n)}.$$

Show that $P_+(V^{\otimes n}) \subset T_+$, where T_+ is the trivial isotypical component in $V^{\otimes n}$ with respect to the action of S_n (the largest submodule of $V^{\otimes n}$ isomorphic to a direct sum of trivial representations of S_n).

(b) Define $P_- : V^{\otimes n} \to V^{\otimes n}$ by

$$P_-(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) u_{\sigma(1)} \otimes u_{\sigma(2)} \ldots \otimes u_{\sigma(n)},$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ. Show that $P_-(V^{\otimes n}) \subset T_-$, where T_- is the isotypical component of the sign representation in $V^{\otimes n}$ (the largest submodule of $V^{\otimes n}$ isomorphic to a direct sum of sign representations of S_n).

(c) Show that $P_+|_{T_+} = \mathrm{id}_{T_+}$, $P_-|_{T_-} = \mathrm{id}_{T_-}$ and deduce that P_+, P_- are projectors onto T_+, T_-.

(d) Show that $T_+ \cong S^n(V)$ and $T_- \cong \wedge^n(V)$.

Exercise 2. (Hilbert's third problem)
This exercise shows how tensor products can be used to solve a problem in 3-dimensional geometry.

(a) Define the Dehn invariant $D(A) \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$ of a polyhedron A by

$$D(A) = \sum_a l(a) \otimes \frac{\beta(a)}{\pi},$$

where the sum is taken over the edges of A, $l(a)$ is the length of a, and $\beta(a)$ is the angle at the edge a. Show that if you cut A into B and C by a straight cut, then $D(A) = D(B) + D(C)$.

(b) Find the angle α at the edge of a regular tetrahedron and prove that it is not a rational multiple of π.
Hint: Assume that $\alpha = \frac{m}{n}\pi$ and deduce that $(x + x^{-1}) = \frac{2}{3}$, where x is a root of unity of degree n. Show by induction that $x^k + x^{-k}$ has denominator 3^k and deduce a contradiction.

(c) Compute the Dehn invariants of a cube and of a regular tetrahedron and conclude that a cube cannot be cut with straight cuts and rebuilt in the shape of a regular tetrahedron of the same volume.