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Problem Set 8 Solutions

Exercise 1. (a) Let G be a finite group, and V_1 and V_k two complex representations, $\mathrm{dim V_1 = 1}$, $\mathrm{dim V_k = k}$. Use characters to show that $V_k \otimes V_1$ is irreducible if and only if V_k is.

(b) Let V be an irreducible complex representation of G of dimension $k > 1$, and suppose that it is the only irreducible representation of G of dimension k. Show that if there is a 1-dimensional complex representation ρ_1 of G and an element $g \in G$ such that $\rho_1(g) \neq 1$, then $\chi_V(g) = 0$. This property is useful in computation of character tables.

Solution 1. (a) Let χ_k be the character of V_k, and χ_1 - the character of V_1. Then $\chi_k\chi_1$ is the character of $V_k \otimes V_1$. Let us compute the inner product

$$(\chi_k\chi_1, \chi_k\chi_1) = \frac{1}{|G|}\sum_g \chi_k(g)\chi_1(g)\overline{\chi_k(g)\chi_1(g)}.$$

Since V_1 is one-dimensional, $\chi_1(g) = \rho_1(g)$ and is a root of unity. Therefore, $\chi_1(g)\overline{\chi_1(g)} = |\chi_1(g)|^2 = 1$ for all $g \in G$. Then we have

$$(\chi_k\chi_1, \chi_k\chi_1) = \frac{1}{|G|}\sum_g \chi_k(g)\overline{\chi_k(g)} = (\chi_k, \chi_k).$$

Therefore, $(\chi_k\chi_1, \chi_k\chi_1) = 1$ if and only if $(\chi_k, \chi_k) = 1$. Equivalently, $V_k \otimes V_1$ is irreducible if and only if V_k is.

(b) By (a) we know that $V \otimes V_1$ is irreducible, and since V is the only irreducible representation of the given dimension, we should have $V \otimes V_1 \simeq V$, and $\chi_V(g)\chi_1(g) = \chi_V(g)$ for all $g \in G$. If $\chi_1(g) \neq 1$ for some $g \in G$, this implies $\chi_V(g) = 0$.

Exercise 2. This exercise shows how to compute the symmetric and exterior powers of linear maps given by explicit matrices.

(a) Let V be a 2-dimensional vector space. Let $f : V \to V$ be given by the matrix

$$f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Find the matrix of $S^2(f) : S^2(V) \to S^2(V)$, where $S^2(V)$ is the second symmetric power of V.

(b) Let U be a 3-dimensional vector space. Let $g : U \to U$ be given the matrix

$$g = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}.$$

Find the matrix of $\wedge^2(g) : \wedge^2(V) \to \wedge^2(V)$, where $\wedge^2(V)$ is the second exterior power of V.

Solution 2. (a) Choose a basis $\{a, b\}$ of V such that $f(a) = pa + rb$ and $f(b) = qa + sb$. Then $\{aa, ab, bb\}$ understood as commuting variables form a basis of S^2V and we have

$$\begin{aligned}
S^2f(aa) &= f(a)f(a) = (pa + rb)(pa + rb) = p^2 \cdot aa + 2pr \cdot ab + r^2 \cdot bb \\
S^2f(ab) &= f(a)f(b) = (pa + rb)(qa + sb) = pq \cdot aa + (ps + rq) \cdot ab + rs \cdot bb \\
S^2f(bb) &= f(b)f(b) = (qa + sb)(qa + sb) = q^2 \cdot aa + 2qs \cdot ab + s^2 \cdot bb.
\end{aligned}$$

It follows that the matrix of S^2f is

$$\begin{pmatrix} p^2 & pq & q^2 \\ 2pr & ps + rq & 2qs \\ r^2 & rs & s^2 \end{pmatrix}.$$

(b) Choose a basis $\{a, b, c\}$ of V such that

$$\begin{aligned} g(a) &= ra + ub + xc, \\ g(b) &= sa + vb + yc, \\ g(c) &= ta + wb + zc. \end{aligned}$$

Then $\{a \wedge b, a \wedge c, b \wedge c\}$ is a basis of $\bigwedge^2 V$ and we have

$$\begin{aligned} \bigwedge^2 g(a \wedge b) = g(a) \wedge g(b) &= (ra + ub + xc)(sa + vb + yc) \\ &= (rv - us) \cdot a \wedge b + (ry - xs) \cdot a \wedge c + (uy - xv) \cdot b \wedge c, \\ \bigwedge^2 g(a \wedge c) = g(a) \wedge g(c) &= (ra + ub + xc)(ta + wb + zc) \\ &= (rw - ut) \cdot a \wedge b + (rz - xt) \cdot a \wedge c + (uz - xw) \cdot b \wedge c, \\ \bigwedge^2 g(b \wedge c) = g(b) \wedge g(c) &= (sa + vb + yc)(ta + wb + zc) \\ &= (sw - vt) \cdot a \wedge b + (sz - yt) \cdot a \wedge c + (vz - yw) \cdot b \wedge c. \end{aligned}$$

It follows that the matrix of $\bigwedge^2 g$ is

$$\begin{pmatrix} rv - us & rw - ut & sw - vt \\ ry - xs & rz - xt & sz - yt \\ uy - xv & uz - xw & vz - yw \end{pmatrix}.$$

Exercise 3. Let $V \simeq \mathbb{C}^n$ and $A : V \to V$ be a linear map with eigenvalues $\{\lambda_i\}_{i=1}^n$. Consider the linear maps $S^2(A) : S^2(V) \to S^2(V)$ and $\wedge^2 A : \wedge^2 V \to \wedge^2 V$. This exercise expresses the trace of a symmetric and exterior square of a linear map in terms of traces in V.

(a) Express the trace $\mathrm{tr}(S^2(A))$ in terms of $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$.

(b) Express the trace $\mathrm{tr}(\wedge^2(A))$ in terms of $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$.

(c) Let V be a representation of a finite group G, $\dim(V) \geq 2$ and let $g \in G$. Use (a) and (b) to express the characters of the representations S^2V and $\wedge^2 V$ in terms of $\chi_V(g)$ and $\chi_V(g^2)$.

Solution 3. Let $\{v_1, \ldots, v_n\}$ be a basis of V such that v_i is a generalized eigenvector associated to λ_i for all $i = 1, \ldots, n$ and A is in Jordan normal form with respect to this basis, i.e. $Av_i = \lambda_i v_i + \mu_i v_{i-1}$ with $\mu_i \in \{0, 1\}$.

(a) The vectors $\{v_i v_j \mid 1 \leq i \leq j \leq n\}$ form a basis of $S^2 V$ and we have $S^2 A(v_i v_j) = (Av_i)(Av_j) = \lambda_i \lambda_j \cdot v_i v_j + \mu_i \lambda_j \cdot v_{i-1} v_j + \lambda_i \mu_j \cdot v_i v_{j-1} + \mu_i \mu_j \cdot v_{i-1} v_{j-1}$, so $\{\lambda_i \lambda_j \mid 1 \leq i \leq j \leq n\}$ is the (multi)set of eigenvalues of $S^2 A$. As the trace of $S^2 A$ is the sum of all eigenvalues of $S^2 A$, we obtain

$$\mathrm{tr}(S^2 A) = \sum_{1 \leq i \leq j \leq n} \lambda_i \lambda_j = \frac{1}{2} \cdot \Big(\sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \lambda_i \lambda_j + \sum_{1 \leq i \leq n} \lambda_i^2 \Big) = \frac{1}{2} \cdot \big(\mathrm{tr}(A)^2 + \mathrm{tr}(A^2) \big),$$

where we also use the fact that the eigenvalues of A^2 are $\{\lambda_i^2 \mid i = 1, \ldots, n\}$.

(b) The vectors $\{v_i \wedge v_j \mid 1 \leq i < j \leq n\}$ form a basis of $\bigwedge^2 V$ and we have $\bigwedge^2 A(v_i \wedge v_j) = (Av_i) \wedge (Av_j) = \lambda_i \lambda_j \cdot v_i \wedge v_j + \mu_i \lambda_j \cdot v_{i-1} \wedge v_j + \lambda_i \mu_j \cdot v_i \wedge v_{j-1} + \mu_i \mu_j \cdot v_{i-1} \wedge v_{j-1}$, so $\{\lambda_i \lambda_j \mid 1 \leq i < j \leq n\}$ is the (multi)set of eigenvalues of $\bigwedge^2 A$. As the trace of $\bigwedge^2 A$ is the sum of all eigenvalues of $\bigwedge^2 A$, we obtain

$$\mathrm{tr}(\bigwedge^2 A) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \frac{1}{2} \cdot \Big(\sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i \leq n} \lambda_i^2 \Big) = \frac{1}{2} \cdot \big(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2) \big),$$

as in part (a).

(c) From (a) and (b) we have

$$\begin{aligned} \chi_{S^2 V}(g) &= \frac{1}{2} \left((\chi_V(g))^2 + \chi_V(g^2) \right), \\ \chi_{\wedge^2 V}(g) &= \frac{1}{2} \left((\chi_V(g))^2 - \chi_V(g^2) \right). \end{aligned}$$

In particular, we have $\chi_V(g^2) = \chi_{S^2 V}(g) - \chi_{\wedge^2 V}(g)$.

Exercise 4. Let V be an n-dimensional complex vector space. Then $GL(V)$ acts in the space $\wedge^m(V)$ by $g \cdot (v_1 \wedge v_2 \wedge \ldots \wedge v_m) = gv_1 \wedge gv_2 \wedge \ldots gv_m$, where $m \leq n$, and on the space $S^k(V)$ by $g \cdot (u_1 u_2 \ldots u_k) = (gu_1)(gu_2)...(gu_k)$.

(a) Show that $\wedge^m(V)$ is an irreducible representation of $GL(V)$, $m \leq n$. *Hint:* Let $\{v_i\}_{i=1}^n$ be a basis in V. Find an element $H \in GL(V)$ such that $\wedge^m H$ is a diagonal operator with all distinct eigenvalues in $\wedge^m(V)$. Then any subrepresentation $W \subset \wedge^m(V)$ should contain a subset of eigenvectors of H. Use an element $P \in GL(V)$ that permutes the basis $\{v_i\}$ of V to conclude that $W = \wedge^m(V)$.

(b) Show that $S^2(V)$ is an irreducible representation of $GL(V)$.

Solution 4. (a) Let H be the diagonal matrix in the basis $\{v_i\}_{i=1}^n$ with the diagonal elements given by distinct prime numbers $p_1, \ldots p_n$. A basis in $\wedge^m(V)$ is given by the set $\{v_{i_1} \wedge \ldots \wedge v_{i_m}\}$ where $1 \leq i_1 < i_2 < \ldots < i_m \leq n$. The operator $\wedge^m H$ is diagonal in this basis with the eigenvalues given by the products of m distinct primes:

$$\wedge^m H(v_{i_1} \wedge \ldots \wedge v_{i_m}) = (p_{i_1} p_{i_2} \ldots p_{i_m}) v_{i_1} \wedge \ldots \wedge v_{i_m}.$$

Since p_i are all distinct primes, these eigenvalues are all distinct. Suppose $W \subset \wedge^m(V)$ is a subrepresentation of the group $GL(V)$. Then $HW \subset W$, which means that W is spanned by a subset of the eigenvectors of H. Suppose that $v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_m} \in W$, and consider another element of the basis of $\wedge^m(V)$, say $v_{j_1} \wedge v_{j_2} \wedge \ldots \wedge v_{j_m}$. Let σ_m be the permutation that sends $v_{i_k} \to v_{j_k}$ for all $k = 1, \ldots m \leq n$. Extend it in an arbitrary way to a permutation $\sigma \in S_n$. Viewed as a permutation of the basis $\{v_i\}_{i=1}^n$ in V, σ is a linear operator given by an invertible matrix $P_\sigma \in GL(V)$. Since W is invariant under $P \in GL(V)$, we have

$$P(v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_m}) = v_{j_1} \wedge v_{j_2} \wedge \ldots \wedge v_{j_m} \in W.$$

Therefore, all elements of the basis of $\wedge^m(V)$ are in W, and thus $\wedge^m(V)$ is an irreducible representation of $GL(V)$.

(b) A basis in $S^2(V)$ is given by the elements $v_{i_1} v_{i_2}$, where $1 \leq i_1 \leq i_2 \leq n$. Similarly to (a) we have the operator H acting diagonally on this basis with all distinct eigenvalues:

$$S^2 H(v_{i_1} v_{i_2}) = (p_{i_1} p_{i_2}) v_{i_1} v_{i_2}.$$

Now the prime numbers in the product can repeat, but the product is still unique to each basis element. Suppose that $W \subset S^2(V)$ is a subrepresentation of $GL(V)$. Then it is spanned by a subset of the eigenvectors of H.

Suppose first that W contains a vector $v_{i_1} v_{i_2}$, where $1 \leq i_1 < i_2 \leq n$. For any $1 \leq j_1 < j_2 \leq n$ consider the permutation $i_1 \to j_1$, $i_2 \to j_2$, completed to a permutation $\sigma \in S_n$. Viewed as a permutation of the basis $\{v_i\}_{i=1}^n$ in V, σ gives rise to a linear operator given by the invertible matrix $P_\sigma \in GL(V)$. Since W is $GL(V)$-invariant, we get

$$P_\sigma(v_{i_1} v_{i_2}) = v_{j_1} v_{j_2} \in W$$

for any $1 \leq j_1 < j_2 \leq n$. Now consider the operator $M = \mathrm{Id} + E_{i_1, i_2} \in GL(V)$, where $E_{i_1, i_2}(v_{i_1}) = v_{i_2}$, and $E_{i_1, i_2} v_j = 0$ for $j \neq i_1$. We have

$$M(v_{i_1} v_{i_2}) = (v_{i_1} + v_{i_2}) v_{i_2} = v_{i_1} v_{i_2} + v_{i_2}^2 \in W.$$

As $v_{i_1} v_{i_2} \in W$, we have that $v_{i_2}^2 \in W$. Now applying a suitable permutation operator that sends $v_{i_2} \to v_k$, we get that $v_k^2 \in W$ for all $1 \leq k \leq n$. Finally, we have that $v_{j_1} v_{j_2} \in W$ for all $1 \leq j_1 \leq j_2 \leq n$, which implies that $W = S^2 V$ and $S^2 V$ is irreducible.

Consider now the case when W contains a vector of the form v_i^2 with $1 \leq i \leq n$. Applying a suitable permutation operator from $GL(V)$, we obtain as before that v_j^2, for any $1 \leq j \leq n$ is also in W. Now let $M = \mathrm{Id} + E_{ij}$, where $E_{ij}(v_i) = v_j$ and $E_{ij}(v_k) = 0$ for $k \neq i$. Then we have $M \in GL(V)$, and

$$M(v_i^2) = (v_i + v_j)^2 = v_i^2 + 2 v_i v_j + v_j^2 \in W.$$

Since we already know that all elements of the form v_j^2, $1 \leq j \leq n$, are in W, we obtain that $v_i v_j \in W$ as well, and so finally all $v_{i_1} v_{i_2} \in W$, where $1 \leq i_1 \leq i_2 \leq n$. Therefore, in this case as well $W = S^2 V$ and $S^2 V$ is irreducible.

Remark: In fact the following more general statement holds: $S^m(V)$ is an irreducible representation of $GL(V)$ for any m. The proof of this fact uses a similar method but requires a more careful construction of operators in $GL(V)$ in order to obtain all elements of the standard basis starting from one of them. The case of $\wedge^m(V)$ is easier because every element of the basis of V can occur at most once as a factor in a basis element of $\wedge^m(V)$.