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## Problem Set 8

**Exercise 1.** (a) Let $G$ be a finite group, and $V_1$ and $V_k$ two complex representations, $\mathrm{dim V_1 = 1}$, $\mathrm{dim V_k = k}$. Use characters to show that $V_k \otimes V_1$ is irreducible if and only if $V_k$ is.

(b) Let $V$ be an irreducible complex representation of $G$ of dimension $k > 1$, and suppose that it is the only irreducible representation of $G$ of dimension $k$. Show that if there is a 1-dimensional complex representation $\rho_1$ of $G$ and an element $g \in G$ such that $\rho_1(g) \neq 1$, then $\chi_V(g) = 0$. This property is useful in computation of character tables.

**Exercise 2.** This exercise shows how to compute the symmetric and exterior powers of linear maps given by explicit matrices.

(a) Let $V$ be a 2-dimensional vector space. Let $f : V \to V$ be given by the matrix

$$f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Find the matrix of $S^2(f) : S^2(V) \to S^2(V)$, where $S^2(V)$ is the second symmetric power of $V$.

(b) Let $U$ be a 3-dimensional vector space. Let $g : U \to U$ be given the matrix

$$g = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}.$$

Find the matrix of $\wedge^2(g) : \wedge^2(V) \to \wedge^2(V)$, where $\wedge^2(V)$ is the second exterior power of $V$.

**Exercise 3.** Let $V \simeq \mathbb{C}^n$ and $A : V \to V$ be a linear map with eigenvalues $\{\lambda_i\}_{i=1}^n$. Consider the linear maps $S^2(A) : S^2(V) \to S^2(V)$ and $\wedge^2 A : \wedge^2 V \to \wedge^2 V$. This exercise expresses the trace of a symmetric and exterior square of a linear map in terms of traces in $V$.

(a) Express the trace $\mathrm{tr}(S^2(A))$ in terms of $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$.

(b) Express the trace $\mathrm{tr}(\wedge^2(A))$ in terms of $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$.

(c) Let $V$ be a representation of a finite group $G$, $\dim(V) \geq 2$ and let $g \in G$. Use (a) and (b) to express the characters of the representations $S^2V$ and $\wedge^2 V$ in terms of $\chi_V(g)$ and $\chi_V(g^2)$.

**Exercise 4.** Let $V$ be an $n$-dimensional complex vector space. Then $GL(V)$ acts in the space $\wedge^m(V)$ by $g \cdot (v_1 \wedge v_2 \wedge \ldots \wedge v_m) = gv_1 \wedge gv_2 \wedge \ldots gv_m$, where $m \leq n$, and on the space $S^k(V)$ by $g \cdot (u_1 u_2 \ldots u_k) = (gu_1)(gu_2)\ldots(gu_k)$.

(a) Show that $\wedge^m(V)$ is an irreducible representation of $GL(V)$, $m \leq n$. *Hint:* Let $\{v_i\}_{i=1}^n$ be a basis in $V$. Find an element $H \in GL(V)$ such that $\wedge^m H$ is a diagonal operator with all distinct eigenvalues in $\wedge^m(V)$. Then any subrepresentation $W \subset \wedge^m(V)$ should contain a subset of eigenvectors of $H$. Use an element $P \in GL(V)$ that permutes the basis $\{v_i\}$ of $V$ to conclude that $W = \wedge^m(V)$.

(b) Show that $S^2(V)$ is an irreducible representation of $GL(V)$.