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Problem Set 6 Solutions

Exercise 1. Consider the group algebra $A = \mathbb{C}[S_3]$ of the group of permutations of 3 elements.

(a) Show that $A \simeq \mathbb{C}[D_3]$, where $D_3 = \{s, r : s^2 = 1, r^3 = 1, srs = r^{-1}\}$ is the dihedral group of order 6.

(b) Classify the one-dimensional irreducible representations of A up to equivalence.

(c) Classify the two-dimensional irreducible representations of A up to equivalence.

(d) Use the obtained classifications and the theorem on the structure of finite dimensional algebras to show that A is a semisimple algebra (without use of Maschke's theorem).

Solution 1. (a) It is easy to check that the linear map $\phi : A \to \mathbb{K}[D_3]$ such that $\phi(s_1) = s, \phi(s_2) = sr$ is an algebra isomorphism.

(b) In dimension 1 we have, $\rho(s)^2 = 1$, therefore $\rho(s) = \pm 1$. Also, $\rho(r)^3 = 1$, so $\rho(r)$ is a third root of unity. However, $\rho(srs) = \rho(r) = \rho(r)^{-1}$, therefore $\rho(r) = 1$. Then we have just two inequivalent representations: the trivial one ρ_{11}, and $\rho_{12}(s) = -1$, $\rho_{12}(r) = 1$.

(c) Because of the isomorphism with the dihedral group, we have a representation by symmetries of a regular triangle, where r acts by rotation by $2\pi/3$ and s by reflection with respect to an axis passing through the origin. It is clearly irreducible, because no linear combination of standard basis vectors is stable under both transformations. Now, by Density theorem, A surjects onto $\oplus_i \mathrm{End}(\mathrm{V_i})$, where V_i are the inequivalent irreducible representations. The dimension of A is 6, and the dimension of $\oplus_i \mathrm{End}(\mathrm{V_i})$ for the representations that we already found, is $1 + 1 + 4$. Therefore, this is the only two-dimensional irreducible representation of A up to equivalence.

(d) We have $A/\mathrm{Rad(A)} \simeq \oplus_\mathrm{i}\mathrm{End}(\mathrm{V_i})$. As $\dim(\mathrm{A}) = \dim(\oplus_\mathrm{i}\mathrm{End}(\mathrm{V_i}))$, the radical is zero, and A is semisimple.

Exercise 2. (a) Let V_1 and V_2 be two-dimensional complex vector spaces with bases $\{x_1, x_2\}$ and $\{y_1, y_2\}$ respectively. Let $A : V_1 \to V_1$ be the linear map given in the basis $\{x_1, x_2\}$ by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $B : V_2 \to V_2$ the linear map given in the basis $\{y_1, y_2\}$ by the matrix

$$B = \begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

The linear map $A \otimes B$ is defined as follows: $(A \otimes B)(v_1 \otimes v_2) = A(v_1) \otimes B(v_2)$. Compute the matrix $A \otimes B$ in the basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2\}$.

(b) Apply the above to find the matrices of the representation $\rho \otimes \rho$ of the group $D_4 = \langle s, r \mid s^2 = 1, r^4 = 1, srs = r^{-1} \rangle$, where ρ is the unique irreducible 2-dimensional representation:

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Derive the decomposition of $\rho \otimes \rho$ into a direct sum of irreducible components.

Solution 2. (a) We will write $(t,p)^T$ for a vector with components t and p in the given basis in V_1 or V_2. We have

$$(A\otimes B)(x_1\otimes y_1)=(A\otimes B)((1,0)^T\otimes(1,0)^T)=A(1,0)^T\otimes B(1,0)^T=(a,c)^T\otimes(s,u)^T=$$

$$=(ax_1+cx_2)\otimes(sy_1+uy_2)=as(x_1\otimes y_1)+au(x_1\otimes y_2)+cs(x_2\otimes y_1)+cu(x_2\otimes y_2)=(as,au,cs,cu)^T.$$

This is the expression of $(A\otimes B)(x_1\otimes y_1)$ in the basis $\{x_1\otimes y_1, x_1\otimes y_2, x_2\otimes y_1, x_2\otimes y_2\}$. Similarly,

$$(A\otimes B)(x_1\otimes y_2)=(A\otimes B)((1,0)^T\otimes(0,1)^T)=A(1,0)^T\otimes B(0,1)^T=(a,c)^T\otimes(t,v)^T=(at,av,ct,cv)^T.$$

$$(A\otimes B)(x_2\otimes y_1)=(A\otimes B)((0,1)^T\otimes(1,0)^T)=A(0,1)^T\otimes B(1,0)^T=(b,d)^T\otimes(s,u)^T=(bs,bu,ds,du)^T.$$

$$(A\otimes B)(x_1\otimes y_2)=(A\otimes B)((0,1)^T\otimes(0,1)^T)=A(0,1)^T\otimes B(0,1)^T=(b,d)^T\otimes(t,v)^T=(bt,bv,dt,dv)^T.$$

So finally the matrix of $A\otimes B$ in the basis $\{x_1\otimes y_1, x_1\otimes y_2, x_2\otimes y_1, x_2\otimes y_2\}$ is given by

$$A\otimes B=\begin{pmatrix} as & at & bs & bt\\ au & av & bu & bv\\ cs & ct & ds & dt\\ cu & cv & du & dv\end{pmatrix}=\begin{pmatrix} aB & bB\\ cB & dB\end{pmatrix}.$$

The same computation generalizes to higher dimensional vector spaces.

(b) Using the given matrices of the two-dimensional irreducible representation $\rho : V\to V$ of D_4 and (a), we easily compute

$$\rho^{\otimes 2}(s)=\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0\end{pmatrix},\qquad \rho^{\otimes 2}(r)=\begin{pmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&-1&0&0\\ 1&0&0&0\end{pmatrix}.$$

The characteristic equation for both matrices is $\lambda^4-2\lambda^2+1=(\lambda^2-1)^2=0$. So the only eigenvalues of both matrices are ±1, which means that the two-dimensional irreducible representation where r acts by rotation by $\pi/2$ does not occur in $\rho^{\otimes 2}$. The matrices are easily diagonalizable and we obtain that $V\otimes V\simeq V_0\oplus V_1\oplus V_2\oplus V_3$.

Remark Of course this decomposition can be easily obtained by computing the character. Use the character table we have computed in class for D_4: $\chi_\rho(1)=2, \chi_\rho(r^2)=-2$, other values of ρ are zeros. Then $\chi_{\rho^{\otimes 2}}(1)=\chi_{\rho^{\otimes 2}}(r^2)=4$, other values are 0. The character table gives the unique decomposition

$$\chi_{\rho^{\otimes 2}}=\chi_0+\chi_1+\chi_2+\chi_3.$$

Exercise 3. Let A, B be finite dimensional algebras. Then $A\otimes B$ is also an algebra, with the multiplication given by $(a_1\otimes b_1)(a_2\otimes b_2)=a_1a_2\otimes b_1b_2$.

(a) Show that $\mathrm{Mat_n}(\mathbb{K})\otimes\mathrm{Mat_m}(\mathbb{K})\simeq\mathrm{Mat_{nm}}(\mathbb{K})$ as associative algebras.

(b) Let V and W be irreducible finite dimensional representations of A and B, respectively. Show that $V\otimes W$ with the action $\rho(a\otimes b)(v\otimes w)=\rho(a)v\otimes\rho(b)w$, is a finite dimensional irreducible representation of $A\otimes B$. *Hint:* To show irreducibility, use the density theorem and (a).

Solution 3. (a) Direct computation. Let E^n_{ij} denote a square $n\times n$ matrix with the only nonzero entry, equal to 1, at the position (i,j), and zeros everywhere else. Then $\phi:\mathrm{Mat_n}(\mathbb{K})\otimes\mathrm{Mat_m}(\mathbb{K})\to\mathrm{Mat_{nm}}(\mathbb{K})$, $\phi(E^n_{ij}\otimes E^m_{lk})=E^{nm}_{mi+l,mj+k}$ respects matrix multiplication:

$$\phi(E^n_{ij}\otimes E^m_{lk})\cdot\phi(E^n_{st}\otimes E^m_{pq})=E^{nm}_{mi+l,mj+k}\cdot E^{nm}_{ms+p,mt+q}=$$

$$=\delta_{js}\delta_{kp}E^{nm}_{mi+l,mt+q}=\phi(\delta_{js}\delta_{kp}(E^n_{it}\otimes E^m_{lq}))=\phi((E^n_{ij}\otimes E^m_{lk})\cdot(E^n_{st}\otimes E^m_{pq})).$$

Extending by bilinearity to $\mathrm{Mat_n}(\mathbb{K})\otimes\mathrm{Mat_m}(\mathbb{K})$ and noticing that $\{E^n_{ij}\}$ form a basis in $\mathrm{Mat_n}(\mathbb{K})$, completes the proof.

(b) The map $\rho(a\otimes b)(v\otimes w)=\rho(a)v\otimes\rho(b)w$ indeed defines a representation of $A\otimes B$: $\rho((a_1\otimes b_1)\cdot(a_2\otimes b_2))(v\otimes w)=\rho(a_1a_2)v\otimes\rho(b_1b_2)w=\rho(a_1)\rho(a_2)v\otimes\rho(b_1)\rho(b_2)w=\rho(a_1\otimes b_1)\rho(a_2\otimes b_2)(v\otimes w)$. SInce V and W are irreducible, by density theorem, the algebra A surjects onto End(V) and the algebra B surjects onto End(W), so $A\otimes B$ surjects onto End(V) $\otimes$ End(W). This space is isomorphic to End(V $\otimes$ W) by (a). Thus, $V\otimes W$ is an irreducible representation of $A\otimes B$.

Remark: If ρ_1 in V and ρ_2 in W are irreducible representations of the same algebra A, then $\rho_1\otimes\rho_2$ in $V\otimes W$, then the surjection argument above fails and the tensor product representation $\rho_1\otimes\rho_2$ in $V\otimes W$ does not have to be irreducible.

Exercise 4. (a) Suppose $H \subset G$ is a normal subgroup of a finite group, and $\rho : G/H \to \mathrm{Aut(V)}$ is a representation of G/H. Let $\phi : G \to G/H$ be the natural surjective homomorphism. Check that $\tilde{\rho} = \rho \circ \phi$ defines a representation of G in V. If ρ is irreducible, show that $\tilde{\rho}$ is irreducible as well. Show that inequivalent representations of G/H lift to inequivalent representations of G.

(b) Let Q_8 denote the group of quaternions, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the defining relations

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2.$$

Find the center $Z(Q_8)$, and describe the structure of $Q_8/Z(Q_8)$. Use (a) to lift the irreducible representations of $Q_8/Z(Q_8)$ to Q_8.

(c) Use the structure theorem of semisimple finite dimensional algebras to find the dimensions of the remaining irreducible representations of Q_8. Use the orthogonality relations to determine their characters.

(d) Use characters to decompose the tensor products of the irreducible representations of Q_8 of dimension > 1 into a direct sum.

Solution 4. (a) If $W' \in V$ is a subrepresentation of $\tilde{\rho}$, then $\tilde{\rho}(g) = \rho(\phi(g))W' \subset W'$ for all $g \in G$. This implies that $\rho(G/H)W' \subset W'$, because $\phi : G \to G/H$ is surjective. So if ρ is irreducible, than $\tilde{\rho}$ is irreducible as well. Now suppose that $A\tilde{\rho}_1(g)A^{-1} = \tilde{\rho}_2(g)$ for all $g \in G$. This implies that $A\rho_1(\phi(t))A^{-1} = \tilde{\rho}_2(t)$ for all $t \in G/H$, because $\phi : G \to G/H$ is surjective. Therefore, inequivalent representations of G/H lift to inequivalent representations of G.

(b) The center $Z(Q_8)$ consists of two elements, ± 1. The quotient $Q_8/Z(Q_8) = \{i, j, k : i^2 = j^2 = k^2 = 1, ij = k\}$. The same group can be presented as $Q_8/Z(Q_8) = \{1, i, j, ij : i^2 = j^2 = 1, ij = ji\}$ This is the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, it is abelian, and it has 4 inequivalent irreducible representations determined by 4 independent choices: $\rho(i) = \pm 1, \rho(j) = \pm 1$. These representations lift to Q_8: $\tilde{\rho}(\pm i) = \rho(i), \tilde{\rho}(\pm j) = \rho(j), \tilde{\rho}(\pm k) = \rho(i)\rho(j)$.

(c) Since $|Q_8| = 8$, taking into account the representations constructed in (b), we have $8 = 1 + 1 + 1 + 1 + 4$, which is the only way 8 breaks up into a sum of squares with 4 ones, and not all ones (otherwise the group would be commutative). Otherwise, you can observe that there are 5 conjugacy classes: 1, -1, $\pm i$, $\pm j$, $\pm k$. Then we have only one representation of dimension 2. Obviously $\chi_2(1) = 2$ and $\chi_2(-1) = -2$. The remaining values of χ_2 can be determined from the orthogonality relations: they need to satisfy $(\chi_2, \chi_i) = \frac{1}{8}\sum_{g \in Q_8} \chi_2(g)\overline{\chi_i(g)} = 0$ for any $\chi_i = \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}$. We have the following table to characters, where $|C_g|$ denotes the number of elements in the conjugacy class.

	1	-1	$\pm i$	$\pm j$	$\pm k$
$\lvert C_g\rvert$	1	1	2	2	2
χ_{11}	1	1	1	1	1
χ_{12}	1	1	-1	1	-1
χ_{13}	1	1	1	-1	-1
χ_{14}	1	1	-1	-1	1
χ_2	2	-2	0	0	0

In fact, the 2-dimensional representation can be realized by the *Pauli matrices* as follows,

$$\rho_2(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_2(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\rho_2(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \rho_2(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(d) The character of $\rho_2 \otimes \rho_2$ is $\chi = \chi_2^2 = (4, 4, 0, 0, 0)$. Taking the inner products, we get

$$(\chi_{1i}, \chi_2^2) = 1/8(4 + 4) = 1, \qquad i = 1, 2, 3, 4$$

$$(\chi_2, \chi_2^2) = 1/8(8 - 8) = 0.$$

Therefore, $\rho_2 \otimes \rho_2 \simeq \rho_{11} \oplus \rho_{12} \oplus \rho_{13} \oplus \rho_{14}$.