October 29, 2024

Problem Set 6

Exercise 1. Consider the group algebra $A = \mathbb{C}[S_3]$ of the group of permutations of 3 elements.

(a) Show that $A \simeq \mathbb{C}[D_3]$, where $D_3 = \{s, r : s^2 = 1, r^3 = 1, srs = r^{-1}\}$ is the dihedral group of order 6.

(b) Classify the one-dimensional irreducible representations of A up to equivalence.

(c) Classify the two-dimensional irreducible representations of A up to equivalence.

(d) Use the obtained classifications and the theorem on the structure of finite dimensional algebras to show that A is a semisimple algebra (without use of Maschke's theorem).

Exercise 2. (a) Let V_1 and V_2 be two-dimensional complex vector spaces with bases $\{x_1, x_2\}$ and $\{y_1, y_2\}$ respectively. Let $A : V_1 \to V_1$ be the linear map given in the basis $\{x_1, x_2\}$ by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $B : V_2 \to V_2$ the linear map given in the basis $\{y_1, y_2\}$ by the matrix

$$B = \begin{pmatrix} s & t \\ u & v \end{pmatrix}.$$

The linear map $A \otimes B$ is defined as follows: $(A \otimes B)(v_1 \otimes v_2) = A(v_1) \otimes B(v_2)$. Compute the matrix $A \otimes B$ in the basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_1, x_2 \otimes y_2\}$.

(b) Apply the above to find the matrices of the representation $\rho \otimes \rho$ of the group $D_4 = \langle s, r \mid s^2 = 1, r^4 = 1, srs = r^{-1} \rangle$, where ρ is the unique irreducible 2-dimensional representation:

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Derive the decomposition of $\rho \otimes \rho$ into a direct sum of irreducible components.

Exercise 3. Let A, B be finite dimensional algebras. Then $A \otimes B$ is also an algebra, with the multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

(a) Show that $\mathrm{Mat_n}(\mathbb{K}) \otimes \mathrm{Mat_m}(\mathbb{K}) \simeq \mathrm{Mat_{nm}}(\mathbb{K})$ as associative algebras.

(b) Let V and W be irreducible finite dimensional representations of A and B, respectively. Show that $V \otimes W$ with the action $\rho(a \otimes b)(v \otimes w) = \rho(a)v \otimes \rho(b)w$, is a finite dimensional irreducible representation of $A \otimes B$. *Hint:* To show irreducibility, use the density theorem and (a).

Exercise 4. (a) Suppose $H \subset G$ is a normal subgroup of a finite group, and $\rho : G/H \to \mathrm{Aut(V)}$ is a representation of G/H. Let $\phi : G \to G/H$ be the natural surjective homomorphism. Check that $\tilde{\rho} = \rho \circ \phi$ defines a representation of G in V. If ρ is irreducible, show that $\tilde{\rho}$ is irreducible as well. Show that inequivalent representations of G/H lift to inequivalent representations of G.

(b) Let Q_8 denote the group of quaternions, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the defining relations

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2.$$

Find the center $Z(Q_8)$, and describe the structure of $Q_8/Z(Q_8)$. Use (a) to lift the irreducible representations of $Q_8/Z(Q_8)$ to Q_8.

(c) Use the structure theorem of semisimple finite dimensional algebras to find the dimensions of the remaining irreducible representations of Q_8. Use the orthogonality relations to determine their characters.

(d) Use characters to decompose the tensor products of the irreducible representations of Q_8 of dimension > 1 into a direct sum.