October 15, 2024

Problem Set 5

Exercise 1. Consider the representation of the group $U(1) = \{e^{i\theta}, \theta \in [0, 2\pi[\} \subset \mathbb{C}$ in $V = \mathbb{C}^2$ given by the rotation matrix

$$\rho(e^{i\theta}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Decompose V into a direct sum of two irreducible unitary complex representations of $U(1)$.

Exercise 2. Let A be an associative algebra over a field k. For a representation V of A, consider the vector space $\mathrm{End}_A(V)$ of endomorphisms of the representation V (linear maps $V \to V$ commuting with the action of A in V). Let V be the left regular representation, $V = A$. Show that $\mathrm{End}_A(A)$ is an associative algebra isomorphic to A^{op}, the algebra A with the opposite multiplication.

Exercise 3. Let $A = \mathrm{Mat}_n(k)$ for a field k. Prove that the algebra A is semisimple, meaning that any finite dimensional representation of A over k is isomorphic to a direct sum of irreducible representations.
Hint: Consider the basis of matrices with a single nonzero matrix element $\{E_{ij}\}$ in A. Show that for a representation V of A, we have $V = \oplus_{i=1}^n E_{ii}V$ and that for $v \in E_{11}V$, the linear span of $\{E_{11}v, E_{21}v, \dots E_{n1}v\}$ is a subrepresentation of V isomorphic to k^n. Conclude by choosing a basis in $E_{11}V$.

Exercise 4. Let A be a finite dimensional algebra, and $\mathrm{Rad}(A)$ the set of all elements of A that act by 0 in all irreducible representations of A.

(a) Show that $\mathrm{Rad}(A)$ is a two-sided ideal in A.

(b) Let $I \subset A$ be a two-sided nilpotent ideal, meaning that there exist $n \in \mathbb{N}$ such that $x^n = 0$ for all $x \in I$. Show that $I \subset \mathrm{Rad}(A)$.

Exercise 5. Recall that the character of a finite dimensional representation V of an algebra A over a field k is defined as $\chi_V(a) = \mathrm{Tr}_V \rho(a)$. Show that if V is a finite dimensional representation of A, and $W \subset V$ a subrepresentation, then the character $\chi_V = \chi_W + \chi_{V/W}$.

Exercise 6. (a) Construct all possible representations of the cyclic group $C_2 = \langle t \mid t^3 = 1\rangle$ in V, where V is a two-dimensional vector space over the field $\mathbb{F}_2$. Decompose the obtained representations into a direct sum of irreducibles.

(b) For the obtained irreducible representations, consider the intertwiners $\phi : V \to V$ that commute with the action of the group C_3. Show how the Schur's lemma fails in the case of the field $\mathbb{F}_2$, which is not algebraically closed.