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Problem Set 4 Solutions

Note: Let A be an associative algebra over k and V a vector space over k . Below we will use the notions of a representation of A in V and of a left A -module V in the same sense, meaning that a representation (V, ρ) gives to the vector space V the structure of a left A -module, and conversely, a structure of a left A -module gives rise to a representation of A in V , namely

$$\rho(a)v = a \cdot v \quad \forall a \in A, v \in V.$$

Exercise 1. (a) Show that the only irreducible finite dimensional representation of the cyclic group C_p with p a prime over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is trivial.

(b) Let V be an irreducible representation of the cyclic group $C_n = \langle t \mid t^n = 1 \rangle$ where $n = p^s m$, $\gcd(p, m) = 1$, $s \in \mathbb{N}$ over $\mathbb{F}_p$. Show that an irreducible representation V of C_n factors through $C_n \rightarrow C_m$, meaning that the operator $\rho(t) : V \rightarrow V$ satisfies $(\rho(t))^m = 1$.

Solution 1. (a) Let V be an irreducible representation of $C_p = \langle t \mid t^p = 1 \rangle$ over $\mathbb{F}_p$:

$$\rho(t) = T : V \rightarrow V.$$

Then $T^p - \text{Id} = (T - \text{Id})^p = 0$ over $\mathbb{F}_p$. Therefore, $(T - \text{Id})$ is an intertwining operator $V \rightarrow V$ of an irreducible representation that is not invertible. By Schur's lemma (over any field) we conclude that $T - \text{Id} = 0$, and $T = \text{Id}$. Since V is irreducible, this means that V is one-dimensional.

(b) Similarly, let V be an irreducible representation of $C_n = \langle t \mid t^n = 1 \rangle$ over $\mathbb{F}_p$. Then

$$T^{p^s m} - \text{Id} = (T^m - \text{Id})^{p^s}$$

over $\mathbb{F}_p$. Therefore, $(T^m - \text{Id}) : V \rightarrow V$ is an intertwining operator of an irreducible representation that is not invertible. By Schur's lemma we deduce that $T^m = \text{Id}_V$. Therefore, the representation factors through the group homomorphism $C_n \rightarrow C_m$.

Exercise 2. Let (V, ρ) be a representation of an associative algebra A over $\mathbb{C}$. A vector $v \in V$ is cyclic if $\rho(A)v = V$ (the vector v generates V as an A -module). A representation admitting a cyclic vector is called cyclic. Show that a representation V is cyclic if and only if it is isomorphic to the representation A/I , where A acts by left multiplication, for a proper left ideal $I \subset A$.

Solution 2. First suppose that V admits a cyclic vector v and define $I = \{a \in A \mid a \cdot v = 0\}$. The linear map $\varphi : A \rightarrow V$ with $\varphi(a) = a \cdot v$ is a homomorphism of A -modules (if we consider A as an A -module via left multiplication) with kernel I . In particular, I is a left ideal in A and $I \subsetneq A$ since $I \cdot v = \{0\} \neq V = A \cdot v$. Now φ is surjective since v is cyclic and we conclude that

$$A/I = A/\ker(\varphi) \cong \text{im}(\varphi) = V.$$

Now suppose that $V \cong A/I$ for some proper left-ideal $I \subset A$ and denote by $\pi : A \rightarrow A/I \cong V$ the canonical (surjective) homomorphism that sends an element to its residue class modulo I . Then $\pi(1)$ is a cyclic vector in V as every element of V is of the form $\pi(a) = \pi(a \cdot 1) = a \cdot \pi(1)$ for some $a \in A$.

Exercise 3. (a) Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Mat}_2(\mathbb{C}).$$

Let L be the *left* ideal, R the *right* ideal and I the *two-sided* ideal in the algebra $\text{Mat}_2(\mathbb{C})$ generated by A . Describe all matrices in the ideals L, R, I .

(b) Classify the left ideals in the matrix algebra $\text{Mat}_n(\mathbb{C})$.

(c) Classify the two-sided ideals in the matrix algebra $\text{Mat}_n(\mathbb{C})$.

(d) Classify the two-sided ideals in the ring $\text{Mat}_2(\mathbb{Z})$.

Solution 3. (a) Let us try to construct the left ideal L generated by A . We have by definition $L = \text{Span}(MA)$, where $M \in \text{Mat}_2(\mathbb{C})$. So we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$$

Linear combinations of such matrices will produce arbitrary matrices with the zero first column. Therefore, L contains all matrices with zeros in the first column and arbitrary elements in the second column.

Similarly, the right ideal R contains all matrices with zeros in the second row and arbitrary elements in the first row.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

For the two-sided ideal, by definition $I = \text{Span}(MAN)$, where $M, N \in \text{Mat}_2(\mathbb{C})$. Then we have

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A_1, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A_2, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = A_3.$$

Using $\mathbb{C}$ -linear combinations of the obtained matrices, we get $aA_2 + bA + cA_3 + dA_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for all $a, b, c, d \in \mathbb{C}$, therefore $I = \text{Mat}_2(\mathbb{C})$. In fact, $\text{Mat}_2(\mathbb{C})$ contains only zero and itself as two-sided ideals.

- (b) Notice that in (a), the rows in the the matrices in the left ideal in $\text{Mat}_2(\mathbb{C})$ span the same subspace in $\mathbb{C}^2$: this is the subspace of vectors of the form $\begin{pmatrix} 0 \\ a \end{pmatrix}$. This holds in general, and follows from matrix multiplication: the product of two matrices MA is a matrix whose rows are linear combinations of the rows of A , as shown here:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \begin{pmatrix} a_{11}R_1 + a_{12}R_2 + \cdots + a_{1n}R_n \\ a_{21}R_1 + a_{22}R_2 + \cdots + a_{2n}R_n \\ \vdots \\ a_{n1}R_1 + a_{n2}R_2 + \cdots + a_{nn}R_n \end{pmatrix}.$$

If you have two matrices with rows in the same subspace of $\mathbb{C}^n$, then their $\mathbb{C}$ -linear combination is also a matrix with rows in the same subspace. Let L be a left ideal in $\text{Mat}_n(\mathbb{C})$, and let us define $V(L)$ to be the vector subspace in $\mathbb{C}^n$ of first rows of matrices in the ideal $L \in \text{Mat}_n(\mathbb{C})$ (it is straightforward to check that it is indeed a vector space). Then we have a well-defined map from the ideals in $\text{Mat}_n(\mathbb{C})$ to subspaces in $\mathbb{C}^n$. Conversely, let $L(V)$ be the ideal of matrices with rows in V (again it is clear that it is a subspace and we checked that it is a left ideal in the computation above). We claim that $V(L(V)) = V$ and $L(V(L)) = L$. Firstly, $V(L(V)) \subseteq V$ as every row of an element of $L(V)$ is an element of V , in particular the first row is. To get that $V(L(V)) \supseteq V$, note that for any $v \in V$, the matrix having only v in the first row is an element of $L(V)$, so that $v \in V(L(V))$. For the other equality, observe that if $M \in L(V(L))$, then every row R_i of M is the first row of some L_i in the ideal L . Recall that E_{ij} is the matrix that has zeros everywhere except for the position (i, j) where the entry is 1. Then, using our computation above, we have that $M = \sum_{i=1}^n E_{i1}L_i$, so that $M \in L$ and $L(V(L)) \subseteq L$. Finally, if $M \in L$, every row R_i is the first row of $E_{i1}M \in L$, proving that $L(V(L)) \supseteq L$ and the claim.

Therefore, there is a 1-1 correspondence between the left ideals in $\text{Mat}_n(\mathbb{C})$ and subspaces in $\mathbb{C}^n$. If we recall that vector subspaces in $\mathbb{C}^n$ are in fact left submodules of $\mathbb{C}^n$ over $\mathbb{C}$, and left ideals in an algebra are 1-1 with the submodules of its left regular representation, then we have a 1-1 correspondence between left $\mathbb{C}$ -submodules of $\mathbb{C}^n$ and left submodules of the regular representation of $\text{Mat}_n(\mathbb{C})$. The set of all complex subspaces of $\mathbb{C}^n$ of dimension m is called the *Grassmannian* $\mathbf{Gr}_m(n)$. So we have that the set of all left ideals in $\text{Mat}_n(\mathbb{C})$ is parametrized by the elements of all Grassmannians $\mathbf{Gr}_m(n)_{0 \leq m \leq n}$.

In particular, if we consider the case $\mathbb{C}^2$, the only nontrivial (different from zero and the whole algebra) left ideals in $\text{Mat}_n(\mathbb{C})$ are in 1-1 correspondence with all 1-dimensional subspaces in $\mathbb{C}^2$, therefore with the elements of the Grassmannian $\text{Gr}(1, 2)$, or the set of all lines in $\mathbb{C}^2$. This set is called the *complex projective space* $\mathbf{P}^1$.

- (c) Suppose that you have an ideal J that contains a matrix with a nonzero entry a_{ij} . Multiplying by the matrix E_{ii} on the left kills all rows except the i th row. Then multiplying by the matrix E_{jj} on the right kills all columns except for the j -th column. Now the resulting matrix has just one nonzero entry a_{ij} , and dividing by a_{ij} we obtain that $E_{ij} \in J$. Moreover, for any $1 \leq k, l \leq n$, multiplying by E_{ki} on the left and E_{jl} on the right, we obtain $E_{ki}E_{ij}E_{jl} = E_{kl} \in J$, because every element of the canonical basis is sent to 0 except e_i which is sent to e_k . Finally by considering linear combinations of these elementary matrices, we can obtain any matrix in $\text{Mat}_n(\mathbb{C})$. Therefore, if there is a matrix with at least one nonzero element in J , then $J = \text{Mat}_n(\mathbb{C})$. Then the only two ideals in $\text{Mat}_n(\mathbb{C})$ are 0 and $\text{Mat}_n(\mathbb{C})$. Recall from the course that such algebras are called *simple*.

(d) The set of two-sided ideals in $\text{Mat}_n(\mathbb{Z})$ is in 1-1 correspondence with $\mathbb{N}$. Recall that the ideals in the ring $\mathbb{Z}$ are principal, for each $m \in \mathbb{N}$ there is an ideal $(m) \subset \mathbb{Z}$ of multiples of m . Let $(m) \subset \mathbb{Z}$ be an ideal and let $J_m = \text{Mat}_n((m))$ be the set of all matrices with entries in (m) . Clearly, J_m is an ideal in $\text{Mat}_n(\mathbb{Z})$. Conversely, let $J \in \text{Mat}_n(\mathbb{Z})$ be a two-sided ideal, and let $I(J) \subset \mathbb{Z}$ be the set of all $(1, 1)$ entries of all matrices in J . Then $I(J)$ is an ideal in $\mathbb{Z}$: if $m, n \in I(J)$ and $r \in \mathbb{Z}$, then we have matrices $M, N \in J$ with $M_{11} = m, N_{11} = n$, so that $rm + n = rM_{11} + N_{11} = (rM + N)_{11} \in I(J)$. By our initial remark, we have $I(J) = (m)$ for some $m \in \mathbb{N}$ and we now claim that $J = J_m$. For any $A \in J$, we have that a_{ij} is the $(1, 1)$ entry of $E_{1i}AE_{j1} \in J$, so that $a_{ij} \in (m)$ and $A \in J_m$. Conversely, to prove that $J_m \subseteq J$, we proceed in the same fashion as in (c). As $m \in (m)$, there is some matrix $A \in J$ such that $A_{11} = m$. Then, not only $E_{11}AE_{11} = mE_{11} \in J$, but also $E_{i1}(mE_{11})E_{1j} = mE_{ij} \in J$. Considering $\mathbb{Z}$ -linear combinations of these matrices, we finally have that $J_m \subseteq J$, finishing the proof of the claim.

Thus there is a 1-1 correspondence between the two-sided ideals in the ring $\text{Mat}_n(\mathbb{Z})$ and the ideals in $\mathbb{Z}$, that are parameterized by $\mathbb{N}$.

Exercise 4. Use the isomorphism $S \cong A/I$ for a maximal left ideal $I \subset A$ to show that a simple module S over $A = \text{Mat}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$.

Solution 4. Set $A := \text{Mat}_n(\mathbb{C})$ and let S be a simple A -module. Then S is cyclic and therefore $S \cong A/I$ for some left ideal I of A by Exercise 2 in PS 4. Note that I has to be a maximal left ideal as S is simple, otherwise there would exist a left ideal J with $I \subsetneq J \subsetneq A$ and J/I would be a proper non-zero submodule of $A/I \cong S$, a contradiction. Now recall from Exercise 3 PS 4 that there is a bijection

$$\{ \text{subspaces of } \mathbb{C}^n \} \xrightarrow{L} \{ \text{left ideals in } A \}$$

that sends a subspace $W \subseteq \mathbb{C}^n$ to the ideal $L(W)$ of matrices with rows in W (this correspondence between left ideals in $\text{Mat}_n(\mathbb{C})$ and subspaces in $\mathbb{C}^n$ is called the *Morita equivalence*). This bijection is inclusion-preserving, so maximal ideals correspond to maximal subspaces and it follows that $I = L(V)$ for some subspace $V \subseteq \mathbb{C}^n$ with $\dim V = n - 1$. Then $\mathbb{C}^n/V$ is one-dimensional, so there exists a linear map $\vartheta: \mathbb{C}^n \rightarrow \mathbb{C}$ with $\ker(\vartheta) = V$. We define a linear map $\varphi: A \rightarrow \mathbb{C}^n$ by

$$M = \begin{pmatrix} \frac{r_1}{r_2} \\ \vdots \\ \frac{r_n}{r_n} \end{pmatrix} \mapsto \begin{pmatrix} \vartheta(r_1) \\ \vartheta(r_2) \\ \vdots \\ \vartheta(r_n) \end{pmatrix},$$

where $r_1, \dots, r_n \in \mathbb{C}^n$ are the rows of the matrix M . Using the definition of matrix multiplication, we see that φ is a homomorphism of A -modules (see also the computation in the solution to part (b) of Exercise 3 PS 4). Furthermore, φ is surjective since ϑ is surjective and $\ker(\varphi) = L(V) = I$ since $\ker(\vartheta) = V$. Using the isomorphism theorem, we conclude that

$$S \cong A/I = A/L(V) = A/\ker(\varphi) \cong \text{im}(\varphi) = \mathbb{C}^n,$$

as required.