

October 8, 2024

Problem Set 4

Note: Let A be an associative algebra over k and V a vector space over k . Below we will use the notions of a representation of A in V and of a left A -module V in the same sense, meaning that a representation (V, ρ) gives to the vector space V the structure of a left A -module, and conversely, a structure of a left A -module gives rise to a representation of A in V , namely

$$\rho(a)v = a \cdot v \quad \forall a \in A, v \in V.$$

Exercise 1. (a) Show that the only irreducible finite dimensional representation of the cyclic group C_p with p a prime over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is trivial.

(b) Let V be an irreducible representation of the cyclic group $C_n = \langle t \mid t^n = 1 \rangle$ where $n = p^s m$, $\gcd(p, m) = 1$, $s \in \mathbb{N}$ over $\mathbb{F}_p$. Show that an irreducible representation V of C_n factors through $C_n \rightarrow C_m$, meaning that the operator $\rho(t) : V \rightarrow V$ satisfies $(\rho(t))^m = 1$.

Exercise 2. Let (V, ρ) be a representation of an associative algebra A over $\mathbb{C}$. A vector $v \in V$ is cyclic if $\rho(A)v = V$ (the vector v generates V as an A -module). A representation admitting a cyclic vector is called cyclic. Show that a representation V is cyclic if and only if it is isomorphic to the representation A/I , where A acts by left multiplication, for a proper left ideal $I \subset A$.

Exercise 3. (a) Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Mat}_2(\mathbb{C}).$$

Let L be the *left* ideal, R the *right* ideal and I the *two-sided* ideal in the algebra $\text{Mat}_2(\mathbb{C})$ generated by A . Describe all matrices in the ideals L, R, I .

(b) Classify the left ideals in the matrix algebra $\text{Mat}_n(\mathbb{C})$.

(c) Classify the two-sided ideals in the matrix algebra $\text{Mat}_n(\mathbb{C})$.

(d) Classify the two-sided ideals in the ring $\text{Mat}_2(\mathbb{Z})$.

Exercise 4. Use the isomorphism $S \cong A/I$ for a maximal left ideal $I \subset A$ to show that a simple module S over $A = \text{Mat}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$.