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Problem Set 3 Solutions

Exercise 1. Let $\rho : G \to GL(1, \mathbb{C}) = \mathbb{C}^*$ be a representation of a finite group G over $\mathbb{C}$. Show that $\|\rho(g)\| = 1$, $\forall\, g \in G$, where $\|\cdot\|$ is the usual norm on $\mathbb{C}$.

Solution 1. Every element $g \in G$ has finite order. Since ρ is a rep and $\|\cdot\|$ is multiplicative, we have $\|\rho(g)^k\| = \|\rho(g)\|^k = 1$, where k is the order of $g \in G$. Therefore, $\|\rho(g)\| = 1$ for all $g \in G$.

Exercise 2. Let G be a finite group acting by permutations on the elements of a basis of a complex vector space V, thus defining a representation of G in V. Show that if $\dim V > 1$, then the representation is not irreducible.

Solution 2. Let $\rho : G \to GL(V)$ be the permutation representation with V having basis $B = \{e_x : x \in X\}$. Let $v = \sum_{x \in X} e_x$. Then for all $g \in G$ we have $\rho(g)v = v$. Therefore $\langle v \rangle$ is a proper non-zero subrepresentation of V, and the representation V is not irreducible.

Exercise 3. Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a 2-dimensional representation of G over $\mathbb{C}$. Suppose that there are two elements g, h of G such that $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

Solution 3. If ρ is not irreducible, it decomposes as a sum of two irreducible subrepresentations of degree 1, say $U_1 = \langle u_1 \rangle$ and $U_2 = \langle u_2 \rangle$. Then $\rho_1(g) = \lambda_1 \in \mathbb{C}^*$ and $\rho_2(g) = \lambda_2 \in \mathbb{C}^*$, similarly $\rho_1(h) = \mu_1 \in \mathbb{C}^*$ and $\rho_2(h) = \mu_2 \in \mathbb{C}^*$. We have

$$\rho(g)\rho(h) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \rho(h)\rho(g),$$

so they commute. This proves that if $\rho(g)$ and $\rho(h)$ do not commute, the two-dimensional representation must be irreducible. Note that we have used Maschke's theorem on complete reducibility of complex representations of finite groups.

Exercise 4. Let $\rho : S_3 \to GL(3, \mathbb{C})$ be the natural representation where the symmetric group S_3 acts by permutations on an orthonormal basis in $\mathbb{C}^3$.

(a) Explicitly find the elements of $\rho(S_3)$.

(b) Decompose ρ as a direct sum of irreducible representations.

(c) Is ρ completely reducible, if we replace $\mathbb{C}$ with a finite field of two or three elements?

Solution 4. (a) Let $\{e_1, e_2, e_3\}$ be a standard basis in $\mathbb{C}^3$, where S_3 acts by permutations. Then we have

$$\rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since the permutations (12) and (23) generate S_3, we can obtain the remaining elements by matrix multiplication:

$$\rho((123)) = \rho((12)) \cdot \rho((23)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\rho((132)) = \rho((23)) \cdot \rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\rho((13)) = \rho((123)) \cdot \rho((12)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

One can easily check that together with $\rho(1) = \mathrm{Id}$ these form a representation of S_3.

(b) Notice that the vector $e_1+e_2+e_3$ spans a subrepresentation V_0 of S_3, where S_3 acts trivially (just as in Ex. 2 above). Notice that $\{e_1-e_2, e_2-e_3\}$ is a basis in an orthogonal complement to V_0, which is invariant under the action of S_3 by Maschke's theorem. The matrices of $\rho_2((12))$ and $\rho_2((23))$ are given in this basis

$$\rho_2((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho_2((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Since these matrices do not commute, by Ex. 3 above we have that $V_2 = \langle e_1-e_2, e_2-e_3\rangle$ is irreducible and $\mathbb{C}^3 = V_0 \oplus V_2$.

(c) The previous decomposition still works over a field $\mathbb{F}_2$ of characteristic 2. The representation V_0 is a direct summand in the permutation representation. Let us consider the action of S_3 in the subspace $V_2(\mathbb{F}_2) = \langle e_1-e_2, e_2-e_3\rangle$. The matrix $\rho((12))$ has the only eigenvalue 1 (also equal to -1 in $\mathbb{F}_2$). If we try to find the eigenvectors, we obtain

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a+b \\ b \end{pmatrix},$$

which implies $b = 0$ in $\mathbb{F}_2$. But the vector $(1,0)^T$ is not invariant under the action of $\rho((23))$:

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

therefore the 2-dimensional representation $V_2(\mathbb{F}_2)$ is irreducible. Thus we get complete reducibility of this particular 3-dimensional representation over $\mathbb{F}_2$. (This does not imply complet reducibility of any representation of S_3 over $\mathbb{F}_2$, in fact we proved in class the converse to Maschke's theorem: if any finite dimensional representation of G is completely reducible, then $\mathrm{char}(k)$ does not divide $|G|$.
The decomposition obtained in (b) does not hold over a field of characteristic 3, since in this case $e_1+e_2+e_3 \in \langle e_1-e_2, e_2-e_3\rangle$. Still, we have that the only one-dimensional subrepresentation of the permutation representation $V_3(\mathbb{F}_3)$ is $V_0 = \langle e_1+e_2+e_3\rangle$. To show that the permutation representation over $\mathbb{F}_3$ it is indecomposable we can consider the action of $\rho((123))$ in $V_3(\mathbb{F}_3)$. The eigenvalues of $\rho((123))$ are third roots of unity, but in $\mathbb{F}_3$ the only solution of $\lambda^3 = 1$ is $\lambda = 1$, so there is only one eigenvalue $\lambda = 1$. To find the eigenvectors we compute

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ a \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

the only solution is $a = b = c$ and the the only eigenvector is $e_1+e_2+e_3$. The matrix of this action is not block-diagonalizable and therefore the representation is indecomposable over $\mathbb{F}_3$ (See the example of the regular representation of the cyclic group C_3 over $\mathbb{F}_3$ that we considered in Lecture 3).

Exercise 5. Let $G = \langle a\rangle$ be a cyclic group of prime order p. Define $\rho: G \to GL(2, \mathbb{F}_p)$ by

$$\rho(a^r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \forall\, 0 \le r \le p-1.$$

(a) Show that ρ is a representation of G over $\mathbb{F}_p$.

(b) Show that ρ is not irreducible.

(c) Show that ρ cannot be decomposed as a direct sum of irreducible representations.

Solution 5. (a) It suffices to define a representation on a generator of the group, and check that the relations hold:

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Longrightarrow \rho(a^r) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \qquad 0 \le r \le p-1$$

so that $\rho(a^r) = \rho(a)^r$ is well defined, and

$$\rho(a^p) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho(1).$$

(b) The 1-dimensional subspace $\langle (1,0)^T\rangle$ is invariant under the action of $\rho(a)$.

(c) Suppose that the given representation is completely reducible. Then it decomposes as a direct sum of two irreducible 1-dimensional representations. We have already found in (b) one invariant subspace spanned by the vector $(1,0)^T$. Let us show that there are no other 1-dimensional invariant subspaces. Since the group is generated by a, it suffices to consider the eigenvectors of $\rho(a)$. The matrix of $\rho(a)$ has a unique eigenvalue 1 and if we solve for the eigenvectors, we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

This implies $b = 0 \pmod{p}$, so we have the only one-dimensional subrepresentation spanned by $(1,0)^T$. Therefore the defined two-dimensional representation over $\mathbb{F}_p$ is indecomposable.

Exercise 6. Let $G = (\mathbb{Z}, +)$, an infinite cyclic group. Define the $\mathbb{C}$-representation $\rho : G \to GL(2, \mathbb{C})$ by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \forall\, n \in \mathbb{Z}.$$

Show that ρ is not completely reducible. (Maschke's Theorem fails for infinite groups).

Solution 6. After checking that the representation is well defined, meaning that $\rho(n+m) = \rho(n)\rho(m)$ for integers n, m, we can consider the question of complete reducibility. Here similarly to Ex. 5 we can look for an invariant subspace of $\rho(1)$, since $1 \in \mathbb{Z}$ generates the additive group $(\mathbb{Z}, +)$ (Recall that $0 \in \mathbb{Z}$ is the neutral element of the additive group $(\mathbb{Z}, +)$, so that $\rho(0) = \mathrm{Id}$ and not $\rho(1)$). We have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore $b = 0$ and $(1,0)^T$ generates the only invariant 1-dimensional subspace. Therefore the representation is indecomposable and not completely reducible.