October 1, 2024

Problem Set 3

Exercise 1. Let $\rho : G \to GL(1, \mathbb{C}) = \mathbb{C}^*$ be a representation of a finite group G over $\mathbb{C}$. Show that $\|\rho(g)\| = 1$, $\forall\, g \in G$, where $\|\cdot\|$ is the usual norm on $\mathbb{C}$.

Exercise 2. Let G be a finite group acting by permutations on the elements of a basis of a complex vector space V, thus defining a representation of G in V. Show that if $\dim V > 1$, then the representation is not irreducible.

Exercise 3. Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a 2-dimensional representation of G over $\mathbb{C}$. Suppose that there are two elements g, h of G such that $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

Exercise 4. Let $\rho : S_3 \to GL(3, \mathbb{C})$ be the natural representation where the symmetric group S_3 acts by permutations on an orthonormal basis in $\mathbb{C}^3$.

(a) Explicitly find the elements of $\rho(S_3)$.

(b) Decompose ρ as a direct sum of irreducible representations.

(c) Is ρ completely reducible, if we replace $\mathbb{C}$ with a finite field of two or three elements?

Exercise 5. Let $G = \langle a \rangle$ be a cyclic group of prime order p. Define $\rho : G \to GL(2, \mathbb{F}_p)$ by

$$\rho(a^r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \forall\, 0 \leq r \leq p-1.$$

(a) Show that ρ is a representation of G over $\mathbb{F}_p$.

(b) Show that ρ is not irreducible.

(c) Show that ρ cannot be decomposed as a direct sum of irreducible representations.

Exercise 6. Let $G = (\mathbb{Z}, +)$, an infinite cyclic group. Define the $\mathbb{C}$-representation $\rho : G \to GL(2, \mathbb{C})$ by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \forall\, n \in \mathbb{Z}.$$

Show that ρ is not completely reducible. (Maschke's Theorem fails for infinite groups).