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Problem Set 2 Solutions

Exercise 1. Let G be a finite group, and consider the group algebra $\mathbb{C}[G]$.

(a) Show that the dimension of an irreducible representation of $\mathbb{C}[G]$ cannot be bigger than $|G|$.

(b) If $G \neq \{1\}$, show that the dimension of an irreducible representation of $\mathbb{C}[G]$ cannot be bigger than $|G| - 1$.

Solution 1. (a) Let $v \in V \setminus \{0\}$ be a vector in an irreducible representation of $\mathbb{C}[G]$. Then $\mathbb{C}[G]v \subset V$ is a subrepresentation of V of dimension $\leq |G|$. As the representation is irreducible, we have $\mathbb{C}[G]v = V$ and $\dim V = \dim \mathbb{C}[G]v \leq |G|$.

(b) If G is nontrivial, and $\dim V = |G|$, we can again pick any $v \in V \setminus \{0\}$, and obtain that $\mathbb{C}[G]v = V$, and this time $g \mapsto gv$ defines an isomorphism of representations. Consider the 1-dimensional subspace generated by $(\sum_{g \in G} g)v$, which is nonzero by our isomorphism. It is a nontrivial subrepresentation, as it is invariant with respect to the action of $\mathbb{C}[G]$: $h(\sum_{g \in G} g)v = (\sum_{g \in G} g)v$. Therefore an irreducible representation cannot have dimension greater than $|G| - 1$.

Exercise 2. Consider the groups D_n given by generators and relations as follows:

$$D_n = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^n = 1 \rangle.$$

(a) Classify the 1-dimensional complex representations of D_n up to isomorphism (the answer depends on the parity of n).

(b) For $k \in \{0, \dots, n-1\}$, consider the following maps:

$$\rho_k(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_k(s_2) = \begin{pmatrix} 0 & \omega^{-k} \\ \omega^k & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/n}$. Find for which values of k the map ρ_k defines an irreducible representation of $\mathbb{C}[D_n]$.

(c) Find the number of non-isomorphic irreducible representations ρ_k. The answer depends on the parity of n.

Solution 2. (a) For n odd there are 2 representations up to isomorphism with $\rho(s_1) = \rho(s_2) = \pm 1$; for n even there are 4 with $\rho(s_1) = \pm 1$ and $\rho(s_2) = \pm 1$. As s_1 and s_2 are generators, we only need to choose $\rho(s_1)$ and $\rho(s_2)$ in $\mathbb{C}$ so that the relations are also satisfied there. Firstly, $\rho(s_1)^2 = \rho(s_2)^2 = 1$ implies that $\rho(s_1) = \pm 1$ and $\rho(s_2) = \pm 1$, giving us $2 \cdot 2 = 4$ initial choices. The last relation needs a distinction between n odd and n even. If n is even, $(\rho(s_1)\rho(s_2))^n = 1$ is always satisfied so the 4 choices all work. If n is odd, $(\rho(s_1)\rho(s_2))^n = 1$ is equivalent to $\rho(s_1) = \rho(s_2)$, leaving us with 2 choices. To see that they are all pairwise non-isomorphic, assume that $\rho_1 \simeq \rho_2$, via $\varphi(v) = \lambda v$ for some $\lambda \neq 0$. Then, for $g \in D_n$, $\varphi(\rho_1(g)) = \rho_2(g)\varphi(1)$ implies that $\lambda\rho_1(g) = \lambda\rho_2(g)$, so that $\rho_1(g) = \rho_2(g)$. So different representations are non-isomorphic in dimension 1, which was exactly what we were looking for.

(b) For n odd, we need that $k \neq 0$; for n even, we need that $k \neq 0, \frac{n}{2}$. As these representations are of dimension 2, they are irreducible if and only if there is no 1-dimensional subspace invariant with respect to $\rho_k(s_1)$ and $\rho_k(s_2)$. However, it is straightforward to see that the matrix $\rho_k(s_1)$ has eigenspaces spanned by $e_1 + e_2$ and $e_1 - e_2$ (for the eigenvalues 1 and -1, respectively), while the matrix $\rho_k(s_2)$ has eigenspaces spanned by $e_1 + \omega^k e_2$ and $e_1 - \omega^k e_2$ (for the eigenvalues 1 and -1, respectively). Thus, they have an eigenspace in common if and only if $\omega^k \in \{\pm 1\}$. For n odd, this happens if and only if $k = 0$ and for n even, this implies that $k = 0$ or $k = \frac{n}{2}$.

(c) For n odd, there are $\frac{n-1}{2}$ non-isomorphic irreducible representations ρ_k and for n even there are $\frac{n}{2} - 1$. Assume that, for $k \neq l$, we have that $\rho_k \simeq \rho_l$ via the isomorphism φ with matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We will use that $\varphi(\rho_k(g)v) = \rho_l(g)\varphi(v)$ twice to obtain information about k and l. Setting $g = s_1$, we have

$$\begin{pmatrix} b & a \\ d & c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \varphi \circ \rho_k(s_1) = \rho_l(s_1) \circ \varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

so that $a = d$ and $b = c$. With $g = s_2$, we have

$$\begin{pmatrix} \omega^k b & \omega^{-k} a \\ \omega^k d & \omega^{-k} c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & \omega^{-k} \\ \omega^k & 0 \end{pmatrix} = \varphi \circ \rho_k(s_2) = \rho_l(s_2) \circ \varphi$$
$$= \begin{pmatrix} 0 & \omega^{-l} \\ \omega^l & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \omega^{-l} c & \omega^{-l} d \\ \omega^l a & \omega^l b \end{pmatrix},$$

so that $\omega^k d = \omega^l a$ implies that $a(\omega^k - \omega^l) = 0$ and $\omega^k b = \omega^{-l} c$ implies that $b(\omega^k - \omega^{-l}) = 0$. As $\omega^k \neq \omega^l$, we can deduce that $a = d = 0$. But φ is nonsingular, implying that $b \neq 0$ and that $\omega^k = \omega^{-l}$, so that $k + l = n$. Setting $\varphi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we see that indeed $\rho_k \simeq \rho_{n-k}$, and we have pairs of isomorphic representations. As $k = 0$ and $k = \frac{n}{2}$ never give an irreducible representation, we just need to divide the number of irreducible representations in (b) by 2, giving $\frac{n-1}{2}$ for n odd and $\frac{n-2}{2} = \frac{n}{2} - 1$ for n even.

Exercise 3. The integer numbers $\mathbb{Z}$ form a commutative group with respect to addition.

(a) Classify the irreducible finite dimensional complex representations of the group $\mathbb{Z}$.

(b) Does it have finite dimensional indecomposable but not irreducible representations? If so, provide a classification.

Solution 3. (a) $\mathbb{Z}$ is an abelian group, therefore all irreducible representations are one-dimensional. They are defined by $\rho(1) = \lambda$, $\rho(n) = \lambda^n$, where $\lambda \in \mathbb{C}$ is nonzero, because the only condition on λ is that it has to be invertible.

(b) Similarly, any representation in dimension m is determined by $\rho(1) \in \mathrm{GL}(m, \mathbb{C})$, and equivalent representations correspond to conjugate matrices. Thus it is enough to consider the Jordan normal form of $\rho(1)$. Observe that in case $\rho(1)$ consists of more than one Jordan block, it is not indecomposable. Thus, the isomorphism classes of indecomposable representations of dimension m are in bijection with Jordan blocks $J_{\lambda,m}$ with nonzero eigenvalue $\lambda \in \mathbb{C}$.

Exercise 4. For a $\mathbb{C}$-algebra A, its center $Z(A)$ is defined as the set of all elements $z \in A$ that commute with all elements in A:

$$za = az \qquad \forall z \in Z(A), \;\; \forall a \in A.$$

(a) Show that if V is an irreducible finite dimensional representation of A, then any element $z \in Z(A)$ acts in V by multiplication by a scalar $c_V(z)$. Show that $c_V : Z(A) \longrightarrow \mathbb{C}$ is an algebra homomorphism. It is called the *central character* of V.

(b) Show that if V is an idecomposable finite dimensional representation of A, then the operator $\rho(z)$ by which $z \in Z(A)$ acts in V, has only one eigenvalue. This eigenvalue, again denoted $c_V(z)$, is the scalar by which z acts in any irreducible subrepresentation of V.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

Solution 4. Let V be an A-module and denote by $\rho\colon A \to \mathrm{End}(V)$ the corresponding representation (so $a \cdot v = \rho(a)(v)$ for all $a \in A$ and $v \in V$). For $z \in Z(A)$ we have

$$a \cdot \rho(z)(v) = \rho(a) \circ \rho(z)(v) = \rho(az)(v) = \rho(za)(v) = \rho(z) \circ \rho(a)(v) = \rho(z)(a \cdot v)$$

for all $a \in A$ and $v \in V$, so $\rho(z)$ is an endomorphism of A-modules.

(a) Now suppose that V is irreducible and let $z \in Z(A)$. By Schur's Lemma, we have $\rho(z) = c_V(z) \cdot \mathrm{id}_V$ for some $c_V(z) \in \mathbb{C}$ (so z acts on V by multiplication by the scalar $c_V(z)$). For $z, z' \in Z(A)$ and $\lambda \in \mathbb{C}$, we have

$$\big(\lambda \cdot c_V(z) + c_V(z')\big) \cdot \mathrm{id}_V = \lambda \cdot \rho(z) + \rho(z') = \rho(\lambda \cdot z + z') = c_V(\lambda \cdot z + z') \cdot \mathrm{id}_V$$

and

$$c_V(z) \cdot c_V(z') \cdot \mathrm{id}_V = \rho(z) \circ \rho(z') = \rho(zz') = c_V(zz') \cdot \mathrm{id}_V.$$

Hence $\lambda \cdot c_V(z) + c_V(z') = c_V(\lambda \cdot z + z')$ and $c_V(z) \cdot c_V(z') = c_V(zz')$, so $c_V : Z(A) \to \mathbb{C}$ is an algebra homomorphism.

(b) Now suppose that V is indecomposable and, let $z \in Z(A)$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of $\rho(z)$ on V. The generalized λ-eigenspace of $\rho(z)$ on V is

$$V_\lambda = \bigcup_{n \in \mathbb{N}} \ker\big((\rho(z) - \lambda \cdot \mathrm{id}_V)^n\big)$$

and as $\rho(z) - \lambda \cdot \mathrm{id}_V$ is an endomorphisms of A-modules, we conclude that $\ker\left((\rho(z) - \lambda \cdot \mathrm{id}_V)^n\right)$ is an A-submodule of V for all $n \in \mathbb{N}$ and that V_λ is an A-submodule of V. By linear algebra, V decomposes as the direct sum of the generalized eigenspaces of $\rho(z)$ (all of which are A-submodules by the above) and as V is indecomposable, we conclude that $\rho(z)$ has a unique eigenvalue $c_V(z)$ on V. If $W \subseteq V$ is an irreducible submodule then $\rho(z)(W) \subseteq W$ and $\rho(z)|_W$ acts by a scalar $c_W(z)$ on W. In particular, $c_W(z)$ is an eigenvalue of $\rho(z)$ on $W \subseteq V$ and we conclude that $c_W(z) = c_V(z)$.

(c) In the situation of part (b), $\rho(z)$ does not have to be a scalar operator. It can be a block-triangular operator or a Jordan block. For example, recall the case of the polynomial algebra where indecomposable representations were given by Jordan blocks $J_{\lambda,n}$ of eigenvalue λ and size n. Since the algebra $\mathbb{C}[x]$ is commutative, it is equal to its center. Then the central element x acts in the n-dimensional vector space by the Jordan matrix $J_{\lambda,n}$ which is not diagonalizable.

Exercise 5. Use Schur's lemma and the structure theorem of finite abelian groups to prove that an abelian group G has exactly $|G|$ inequivalent complex irreducible representations.

Solution 5. By Schur's Lemma we know that all irreducible representations of an abelian group G are 1-dimensional.

By the structure theorem for finite abelian groups, G is isomorphic to a direct product of cyclic groups, so we can write $G \cong C_{m_1} \times C_{m_2} \times \cdots C_{m_r}$, where C_{m_i} is a cyclic group with a (fixed) generator x_i of order m_i. Then every one-dimensional representation $\rho\colon G \to \mathbb{C}^\times$ is determined by an r-tuple $\lambda(\rho) = (\lambda_1, \dots, \lambda_r)$ with $\lambda_i = \rho(1, \cdots, x_i, \cdots, 1)$. Note that λ_i is an m_i-th root of unity since x_i has order m_i and ρ is a homomorphism. We claim that λ defines a bijection

$$\{\text{ one-dimensional representations of } G\ \} \xrightarrow{\lambda} \left\{(\lambda_1, \dots, \lambda_r) \mid \lambda_i \in \mathbb{C}^\times \text{ is an } m_i\text{-th root of unity}\right\}.$$

Indeed, λ is injective since $x_1, \dots, x_r$ generate G (and therefore any homomorphism $\rho\colon G \to \mathbb{C}^\times$ is uniquely determined by the images $\lambda_1, \dots, \lambda_r \in \mathbb{C}^\times$ of $x_1, \dots, x_r$). Conversely, for an r-tuple $(\lambda_1, \dots, \lambda_r)$ such that λ_i is an m_i-th root of unity, we can define a unique homomorphism $\rho\colon G \to \mathbb{C}^\times$ with $\rho(x_i) = \lambda_i$ for $i = 1, \dots, r$. (This follows from properties of the direct product and the fact that the cyclic group C_{m_i} is isomorphic to the group of m_i-th roots of unity in $\mathbb{C}^\times$.) Note that two one-dimensional representations are equivalent if and only if they are equal because $\mathbb{C}^\times$ is abelian.

Now the set on the right hand side has $m_1 \cdots m_r = |C_{m_1}| \cdots |C_{m_r}| = |G|$ elements, so G has precisely $|G|$ irreducible representations, each of them one-dimensional.