September 24, 2024

Problem Set 2

Exercise 1. Let G be a finite group, and consider the group algebra $\mathbb{C}[G]$.

(a) Show that the dimension of an irreducible representation of $\mathbb{C}[G]$ cannot be bigger than $|G|$.

(b) If $G \neq \{1\}$, show that the dimension of an irreducible representation of $\mathbb{C}[G]$ cannot be bigger than $|G| - 1$.

Exercise 2. Consider the groups D_n given by generators and relations as follows:

$$D_n = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^n = 1 \rangle.$$

(a) Classify the 1-dimensional representations of D_n up to isomorphism (the answer depends on the parity of n).

(b) For $k \in \{0, \dots, n-1\}$, consider the following maps:

$$\rho_k(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_k(s_2) = \begin{pmatrix} 0 & \omega^{-k} \\ \omega^k & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/n}$. Find for which values of k the map ρ_k defines an irreducible representation of $\mathbb{C}[D_n]$.

(c) Find the number of non-isomorphic irreducible representations ρ_k. The answer depends on the parity of n.

Exercise 3. The integer numbers $\mathbb{Z}$ form a commutative group with respect to addition.

(a) Classify the irreducible finite dimensional complex representations of the group $\mathbb{Z}$.

(b) Does it have finite dimensional indecomposable but not irreducible representations? If so, provide a classification.

Exercise 4. For a $\mathbb{C}$-algebra A, its center $Z(A)$ is defined as the set of all elements $z \in A$ that commute with all elements in A:

$$za = az \qquad \forall z \in Z(A), \ \forall a \in A.$$

(a) Show that if V is an irreducible finite dimensional representation of A, then any element $z \in Z(A)$ acts in V by multiplication by a scalar $c_V(z)$. Show that $c_V : Z(A) \longrightarrow \mathbb{C}$ is an algebra homomorphism. It is called the *central character* of V.

(b) Show that if V is an idecomposable finite dimensional representation of A, then the operator $\rho(z)$ by which $z \in Z(A)$ acts in V, has only one eigenvalue. This eigenvalue, again denoted $c_V(z)$, is the scalar by which z acts in any irreducible subrepresentation of V.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

Exercise 5. Use Schur's lemma and the structure theorem of finite abelian groups to prove that an abelian group G has exactly $|G|$ inequivalent complex irreducible representations.