December 17, 2024

Problem Set 13 Solutions

Exercise 1. Let V_λ denote the Specht module for S_n, where λ is a partition of n.

(a) Show that
$$\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda \simeq \bigoplus_{\mu \in R(\lambda)} V_\mu,$$
where $R(\lambda)$ is the set of Young diagrams obtained by removing one square from Y_λ.

(b) Show that
$$\mathrm{Ind}^{S_n}_{S_{n-1}} V_\mu \simeq \bigoplus_{\lambda \in A(\mu)} V_\lambda,$$
where $A(\mu)$ is the set of Young diagrams obtained by adding one square from Y_μ.

Hint: Use the formula for the character of V_μ in (a) and the Frobenius reciprocity in (b).

Exercise 2. (Transitivity of the induction) Let $K \subset H \subset G$ be subgroups of a finite group G and V a complex representation of K. Show that
$$\mathrm{Ind}^G_H \, \mathrm{Ind}^H_K V \simeq \mathrm{Ind}^G_K V.$$

Hint: Use the tensor product form of the induced representations.

Exercise 3. (a) Let G be a finite group and V_R an irreducible representation of G defined over the real numbers. Show that its complexification $V = \mathbb{C} \otimes_{\mathbb{R}} V_R$ is a representation of real type.

(b) Show that all Specht modules V_λ for S_n are of real type.

(c) Use the Frobenius-Schur indicator to find the sum of dimensions of all irreducible representations of S_n.