December 10, 2024

Problem Set 12 Solutions

Exercise 1. Recall that an irreducible Specht module V_λ for S_n is determined by a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_p)$, such that $\sum_{i=1}^p \lambda_i = n$. It can be defined by $V(\lambda) = \mathbb{C}[S_n]c_\lambda$, where $c_\lambda = a_\lambda b_\lambda$ with

$$a_\lambda = \frac{1}{|P_\lambda|}\sum_{g\in P_\lambda} g; \quad b_\lambda = \frac{1}{|Q_\lambda|}\sum_{g\in Q_\lambda}(-1)^g g.$$

Here the subgroups $P_\lambda \in S_n$ and $Q_\lambda \in S_n$ are the stabilizers respectively of rows and columns of a Young tableau T_λ of shape λ.

(a) In class we showed that $c_\lambda^2 = x(\lambda)c_\lambda$, where $x(\lambda) \in \mathbb{Q}$ is a coefficient. Find $x(\lambda)$. *Hint:* Consider the action of c_λ in the regular representation of S_n.

(b) Let $C = \sum_{i<j}(ij) \in \mathbb{C}[S_n]$ be the sum of all transpositions. Show that C acts on the Specht module V_λ by multiplication by the scalar $z(\lambda) = \sum_{j=1}^p \sum_{i=1}^{\lambda_j}(i-j)$. (The integer $z(\lambda)$ is called the *content* of the Young diagram of shape λ.)

Exercise 2. Let $G = SL(2, \mathbb{F}_q)$ be the group of 2×2 matrices of determinant 1 with coefficients in the field of q elements $\mathbb{F}_q$ ($q \geq 3$ a prime). Consider the 2-dimensional $\mathbb{F}_q$-vector space V with the basis $e_1 = (1,0)$ and $e_2 = (0,1)$.

(a) Find the order of G.

(b) Show that G acts transitively on $V \setminus \{(0,0)\}$.

(c) Find the stabilizer $N \subset G$ of e_1.

(d) Show that the representation (F, ρ) of G in the complex vector space F of functions $f : V \setminus \{(0,0)\} \to \mathbb{C}$ given by

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x,y) = f(dx - by, -cx + ay)$$

is isomorphic to the induced representation of G from the trivial representation of N.

(e) Use Frobenius reciprocity to deduce that ρ is not irreducible.

Exercise 3. Let $K \subset G$ be a subgroup, and $\mathbb{C}_\chi$ a one-dimensional representation of K with character $\chi : K \to \mathbb{C}^\star$. Consider the central idempotent corresponding to χ:

$$e_\chi = \frac{1}{|K|}\sum_{g\in K}\chi(g)^{-1}g \quad \in \mathbb{C}[K].$$

Show that the induced representation $\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}\mathbb{C}_\chi$ is naturally isomorphic to $\mathbb{C}[G]e_\chi$, with the action of G in $\mathbb{C}[G]e_\chi$ by left multiplication.