

December 3, 2024

Problem Set 11 Solutions

- Exercise 1.** (a) Let $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of order 6. Describe the irreducible complex representations of D_3 and compute its character table. (Use Ex. 1, PS 6 and Ex. 1, PS 7).
- (b) Decompose ρ_{lreg} the left regular representation $\mathbb{C}[D_3]$ into a direct sum of irreducible representations. Similarly, consider the right regular representation ρ_{rreg} of $\mathbb{C}[D_3]$ by multiplication on the right and decompose it into a direct sum of irreducible representations.
- (c) As an associative algebra $\mathbb{C}[D_3]$ is isomorphic to a direct sum of matrix algebras. This decomposition provides a basis in $\mathbb{C}[D_3]$ given by the matrix elements $\{a_{ij}^V\}_{V \in \text{Irr}}$ of $\text{End}(V)$, which is consistent with the decomposition of ρ_{lreg} and ρ_{rreg} . Express this basis in terms of the basis $\{g\}_{g \in D_3}$.
- (d) For each irreducible V decompose the representation ρ_{ad} of D_3 acting on $\text{End}(V)$ by $\rho_{ad}(g)(f)(v) = \rho_V(g) \circ f(\rho_V(g^{-1})v)$ as a direct sum of irreducible representations. *Hint:* show that $V \simeq V^*$ for all irreducible V of D_3 and use characters.
- (e) Consider the adjoint action of D_3 on $\mathbb{C}[D_3]$: $\rho_{ad}(g)(h) = ghg^{-1}$. Use (d) to decompose ρ_{ad} into a direct sum of irreducible representations.
- (f) Find the center of the algebra $\mathbb{C}[D_3]$.

Solution 1. (a) According to the argument in PS 7, Ex.1 there are three irreducible representations of D_3 : the trivial 1-dimensional V_0 , the 1-dimensional sign representation V_s and the 2-dimensional irreducible representation V_2 given by the symmetries of an equilateral triangle on a plane. Set $\xi = e^{2\pi i/3}$. We have

$$\rho_0(r) = \rho_0(s) = 1, \quad \rho_s(r) = 1, \quad \rho_s(s) = -1, \quad \rho_2(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2(r) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

The conjugacy classes in D_3 are $\{(1), (r, r^2), (s, sr, sr^2)\}$. The character table is given by

	(1)	(r, r ²)	(s, sr, sr ²)
V ₀	1	1	1
V _s	1	1	-1
V ₂	2	-1	0

- (b) By density theorem we have

$$\mathbb{C}[D_3] \simeq \text{End}(V_0) \oplus \text{End}(V_s) \oplus \text{End}(V_2).$$

Each direct summand is a two-sided ideal in $\mathbb{C}[D_3]$. Each $\text{End}(V_i) \simeq V_i^{\oplus d_i}$, where each column of $\text{End}(V_i)$ is a subrepresentation isomorphic to V_i . (see for example Ex. 4, PS 4). Therefore, we have

$$\rho_{lreg} \simeq V_0 \oplus V_s \oplus V_2^{\oplus 2}.$$

Similarly, each $\text{End}(V_i)$ acts on itself by right multiplication and since it is a simple algebra, it decomposes into a direct sum of irreducible representations, all isomorphic to V_i , since V_i is the only irreducible representation of $\text{End}(V_i)$. For the right action, each row of the matrix of $\text{End}(V_i)$ is a subrepresentation, isomorphic to V_i . So we have

$$\rho_{rreg} \simeq V_0 \oplus V_s \oplus V_2^{\oplus 2}.$$

- (c) The algebra $\mathbb{C}[D_3]$ is isomorphic as an associative algebra to the following direct sum of matrix algebras,

$$\mathbb{C}[D_3] \simeq \text{End}(V_0) \oplus \text{End}(V_s) \oplus \text{End}(V_2) \simeq \text{Mat}_1(\mathbb{C}) \oplus \text{Mat}_1(\mathbb{C}) \oplus \text{Mat}_2(\mathbb{C}).$$

Since we know how each matrix algebra decomposes with respect to the left and right regular action of D_3 , it is easy to compute the matrix elements. With respect to the left and right regular actions, the component corresponding

to $\text{End}(V_0)$ is the trivial representation, given by the sum of all group elements, and for the sign representations, the sum of group elements with their signs, where we use the isomorphism $D_3 \simeq S_3$:

$$\text{End}(V_0) = \text{Span} \left(\sum_{g \in D_3} g \right), \quad \text{End}(V_s) = \text{Span} \left(\sum_{g \in S_3} \varepsilon(g)g \right) = \text{Span}(1 + r + r^2 - s - sr - sr^2).$$

To find the matrix elements of $\text{End}(V_2)$ we recall from (b) that the columns and the rows of $\text{End}(V_2)$ are isomorphic to V_2 with respect to the left (resp. right) action of the generators. We can first find the eigenspace of $\rho_{lreg}(r)$ with the eigenvalue ξ :

$$r \cdot v = r \cdot (a + br + cr^2 + ds + esr + fsr^2) = \xi(a + br + cr^2 + ds + esr + fsr^2)$$

therefore $v = a(1 + \xi^2 r + \xi r^2) + d(s + \xi sr + \xi^2 sr^2)$. In a similar way we can find the eigenvector of $\rho_{rreg}(r)$ with eigenvalue ξ :

$$w \cdot r = (a' + b'r + c'r^2 + d's + e'sr + f'sr^2) \cdot r = \xi(a' + b'r + c'r^2 + d's + e'sr + f'sr^2).$$

Therefore, $w = a'(1 + \xi^2 r + \xi^2 r^2) + d'(s + \xi^2 sr + \xi sr^2)$. To have a common eigenvector for $\rho_{lreg}(r)$ and $\rho_{rreg}(r)$ with the same eigenvalue ξ , we must have $d = d' = 0$ and $a = a'$. We can take $a_{11} = (1 + \xi^2 r + \xi r^2)$, which is first element of the matrix of $\text{End}(V_2)$. Note that the action of s swaps two basis elements of V_2 . Then acting by s on the right and on the left of a_{11} , we obtain the next element along the first column $s \cdot a_{11} = a_{21} = (s + \xi^2 sr + \xi sr^2)$ and along the first row $a_{11} \cdot s = (s + \xi sr + \xi^2 sr^2) = a_{12}$. Finally, the common eigenvector of $\rho_{lreg}(r)$ and $\rho_{rreg}(r)$ with eigenvalue ξ^2 is $a_{22} = (1 + \xi r + \xi^2 r^2)$. You can check that $s \cdot a_{12} = a_{22}$ and $a_{21} \cdot s = a_{22}$. Finally we have the following matrix:

$$\text{End}(V_2) = \begin{pmatrix} 1 + \xi^2 r + \xi r^2 & s + \xi sr + \xi^2 sr^2 \\ s + \xi^2 sr + \xi sr^2 & 1 + \xi r + \xi^2 r^2 \end{pmatrix}.$$

Notice that $\{\sum_{g \in D_3} g, \sum_{g \in D_3} \varepsilon(g)g, a_{11}, a_{12}, a_{21}, a_{22}\}$ form a basis in $\mathbb{C}[D_3]$. Moreover, we have that each endomorphism algebra is a two-sided ideal in $\mathbb{C}[D_3]$.

- (d) We have $\chi_{V^*}(g) = \overline{\chi_V(g)}$. Since the characters of all irreducible representations of D_3 are real, each of them is self-dual. Alternatively, we can notice from the character table that $\chi_0^2 = \chi_0$, $\chi_s^2 = \chi_0$ and $\chi_2^2 = \chi_2 + \chi_0 + \chi_s$. Therefore, each $V \otimes V$ contains a trivial representation, and therefore $V \otimes V \simeq V^* \otimes V$, and $V \simeq V^*$. Recall (Lecture 6) that for an irreducible representation V , $\text{End}(V)$ with the action of ρ_{ad} is isomorphic to the representation $V \otimes V^*$. Since in our case $V^* \simeq V$, we have $\text{End}(V) \simeq V \otimes V$ and

$$\text{End}(V_0) \simeq V_0 \otimes V_0 \simeq V_0, \quad \text{End}(V_s) \simeq V_s \otimes V_s \simeq V_0, \quad \text{End}(V_2) \simeq V_2 \otimes V_2 \simeq V_0 \oplus V_s \oplus V_2.$$

- (e) Note that the action of the adjoint representation on $\mathbb{C}[D_3] \simeq \text{End}(V_0) \oplus \text{End}(V_s) \oplus \text{End}(V_2)$ is given by the adjoint action on each of the direct summands, computed in (d). Then we have

$$\rho_{ad} \simeq V_0 \oplus V_0 \oplus V_0 \oplus V_s \oplus V_2 \simeq V_0^{\oplus 3} \oplus V_s \oplus V_2.$$

- (f) Since the action of ρ_{ad} on $\mathbb{C}[D_3]$ is given by $\rho_{ad}(g)h = ghg^{-1}$, we conclude that the center is spanned by the trivial isotypical component of this representation. From (e) we have

$$Z(\mathbb{C}[D_3]) \simeq (\rho_{ad})^G \simeq V_0^{\oplus 3}.$$

Therefore, the center is 3-dimensional. We also know that $\sum_{h \in C} h$ for any conjugacy class $C \subset D_3$ is a central element. Therefore, the center is spanned by

$$\{1, r + r^2, s + sr + sr^2\} = \{1, (123) + (132), (12) + (23) + (13)\},$$

where the second presentation uses the group isomorphism $D_3 \simeq S_3$. We can also notice that the traces of the matrices in the matrix presentation of $\mathbb{C}[D_3]$ computed in (c) provide another basis in the center, namely

$$1 + r + r^2 + s + sr + sr^2, \quad 1 + r + r^2 - s - sr - sr^2, \quad 2 - r - r^2.$$

This basis has the property that the product of any two distinct elements is zero. After a renormalization we can have

$$e_1 = \frac{1}{6}(1 + r + r^2 + s + sr + sr^2), \quad e_2 = \frac{1}{6}(1 + r + r^2 - s - sr - sr^2), \quad e_3 = \frac{1}{3}(2 - r - r^2).$$

Then $e_i e_j = \delta_{ij} e_i$. Central elements with this property are called the *central idempotents*.

Exercise 2. The purpose of this exercise is to illustrate the statements used in the proof of Burnside's theorem (see Lecture 9). Let $G = A_4$, the alternating group of 4 elements.

- (a) We have proved in class that if V is an irreducible representation of G and C a conjugacy class in G such that $\gcd(|C|, \dim(V)) = 1$, then for any $g \in C$ we have either $\chi_V(g) = 0$, or $\rho_V(g) = \lambda \text{Id}_V$. For each nontrivial conjugacy class in A_4 and irreducible representation satisfying the condition $\gcd(|C|, \dim(V)) = 1$, find whether $g \in C$ acts as a scalar in V or has zero character.
- (b) We also proved that if G has a conjugacy class C of a prime power order, then G has a proper nontrivial normal subgroup H defined by $H = \langle ab^{-1}, a, b \in C \rangle \triangleleft G$. Find all conjugacy classes of prime power order in A_4 and construct the corresponding normal subgroups.

Solution 2. (a) We start with the character table of A_4 given for example in Lecture 8. The second line shows the number of elements in each conjugacy class.

	1	(12)(34)	(123)	(132)
$ C $	1	3	4	4
V_0	1	1	1	1
V_ξ	1	1	ξ	ξ^2
V_{ξ^2}	1	1	ξ^2	ξ
V_3	3	-1	0	0

First note that any 1-dimensional representation of G has the property $\gcd(|C|, \dim(V)) = 1$ and that any group element acts by a nonzero scalar given by a root of unity on a 1-dimensional representation, because $(\rho_V(g))^n = \text{Id}$ for $n = \text{order}(g)$. Therefore, for any C , an element $g \in C$ act by a scalar on any 1-dimensional representation.

Now consider V_3 of dimension 3. We have $\gcd(|C|, 3) = 1$ for $C = C_{(123)}$ and $C = C_{(132)}$. Looking at the character table, in both cases we have $\chi_{V_3}((123)) = \chi_{V_3}((132)) = 0$.

- (b) Consider $C_{(12)(34)}$ of order 3. It gives rise to the nontrivial proper normal subgroup

$$H_1 = \langle ab^{-1}, a, b \in \{(12)(34), (13)(24), (14)(23)\} \rangle = \{1, (12)(34), (13)(24), (14)(23)\} = K \triangleleft A_4.$$

Thus we obtain the Klein subgroup which is normal in A_4 . Consider $C_{(123)}$ of order 2². It gives rise to the nontrivial proper normal subgroup

$$H_2 = \langle ab^{-1}, a, b \in \{(123), (214), (341), (432)\} \rangle = \{1, (12)(34), (13)(24), (14)(23)\} = K \triangleleft A_4.$$

Since the products are of the form $(123)(214)^{-1} = (13)(24)$, we obtain the Klein subgroup in A_4 . Similarly, starting from the conjugacy class $C_{(132)}$ of order 2², we obtain the same Klein subgroup $K \triangleleft A_4$.

Remark In fact, because K is the only nontrivial proper normal subgroup in A_4 , we can deduce without a computation that every conjugacy class of prime power order gives rise to the same group K .

Exercise 3. Consider the group $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$ and the subgroups $C_3 = \{1, r, r^2\} \subset D_3$ and $C_2 = \{1, s\} \subset D_3$.

- (a) Use the character formula for the induced representation (Lecture 10) to decompose into the irreducible components the representation $\text{Ind}_{C_3}^{D_3} V$ for each irreducible representation V of C_3 .
- (b) Use the Frobenius reciprocity (Lecture 10) to decompose into the irreducible components the representation $\text{Ind}_{C_2}^{D_3} V$ for each irreducible representation V of C_2 .

Solution 3. (a) Let us first recall the classification of the irreducible representations of $C_3 = \langle r \mid r^3 = 1 \rangle$. There are exactly 3 inequivalent irreducible representations of the cyclic group: V_0, V_ξ, V_{ξ^2} where $\rho(r) = 1, \xi, \xi^2$ respectively. Consider the right cosets with respect to C_3 : $\{\sigma_1 = C_3 1, \sigma_s = C_3 s\}$. For $g \in D_3$, we have $\sigma_i g = \sigma_i$ if and only if g is in the conjugacy classes $\{(1), (r, r^2)\}$. According to the Frobenius character formula for an induced representation $\text{Ind}_{C_3}^{D_3}(V)$ we have

$$\chi(g) = \sum_{\sigma_i: \sigma_i g = \sigma_i} \chi_V(x_\sigma g x_\sigma^{-1}).$$

So we have for $\text{Ind}_{C_3}^{D_3} V$ where $V = V_0, V_\xi, V_{\xi^2}$:

$$\chi(1) = \chi_V(1) + \chi_V(1) = 2, \quad \chi(s) = 0,$$

the second equality follows because the action of s permutes the right C_3 -cosets. If $V = V_0$, we have

$$\chi(r) = \chi_V(r) + \chi_V(srs) = \chi_V(1) + \chi_V(r^{-1}) = 2.$$

If $V = V_\xi$ or $V = V_{\xi^2}$, we have

$$\chi(r) = \chi_V(r) + \chi_V(srs) = \chi_V(1) + \chi_V(r^{-1}) = \xi + \xi^2 = -1.$$

Finally we have the following characters, that we have added as extra lines in the character table of D_3 (see Ex. 1 above):

	(1)	(r, r ²)	(s, sr, sr ²)
V_0	1	1	1
V_s	1	1	-1
V_2	2	-1	0
$\text{Ind}_{C_3}^{D_3} V_0$	2	2	0
$\text{Ind}_{C_3}^{D_3} V_\xi$	2	-1	0
$\text{Ind}_{C_3}^{D_3} V_{\xi^2}$	2	-1	0

Comparing the characters, we conclude

$$\text{Ind}_{C_3}^{D_3} V_0 \simeq V_0 \oplus V_s, \quad \text{Ind}_{C_3}^{D_3} V_\xi \simeq \text{Ind}_{C_3}^{D_3} V_{\xi^2} \simeq V_2.$$

- (b) We will use Frobenius reciprocity. The group $C_2 = \{1, s\}$ has two inequivalent 1-dimensional representations V_0 and V_s , where s acts as ± 1 respectively. We have $\chi_{V_0}(1)\chi_{V_s}(1) = 1$ and $\chi_{V_s}(1) = 1$, $\chi_{V_s}(s) = -1$. Looking at the character table for $S_3 \simeq D_3$, we conclude that

$$\chi_{\text{Res}_{C_2}^{D_3} V_2}(1) = 2, \quad \chi_{\text{Res}_{C_2}^{D_3} V_2}(s) = 0.$$

$$\chi_{\text{Res}_{C_2}^{D_3} V_0}(1) = 1, \quad \chi_{\text{Res}_{C_2}^{D_3} V_0}(s) = 1.$$

$$\chi_{\text{Res}_{C_2}^{D_3} V_s}(1) = 1, \quad \chi_{\text{Res}_{C_2}^{D_3} V_s}(s) = -1.$$

Therefore we conclude that

$$\text{Res}_{C_2}^{D_3} V_2 \simeq V_0 \oplus V_s, \quad \text{Res}_{C_2}^{D_3} V_0 \simeq V_0, \quad \text{Res}_{C_2}^{D_3} V_s \simeq V_s.$$

Using the Frobenius reciprocity we compute

$$\dim \text{Hom}_{D_3}(V_2, \text{Ind}_{C_2}^{D_3} V_0) = \dim \text{Hom}_{C_2}(\text{Res}_{C_2}^{D_3} V_2, V_0) = 1;$$

$$\dim \text{Hom}_{D_3}(V_0, \text{Ind}_{C_2}^{D_3} V_0) = \dim \text{Hom}_{C_2}(\text{Res}_{C_2}^{D_3} V_0, V_0) = 1;$$

$$\dim \text{Hom}_{D_3}(V_s, \text{Ind}_{C_2}^{D_3} V_0) = \dim \text{Hom}_{C_2}(\text{Res}_{C_2}^{D_3} V_s, V_0) = 0.$$

Therefore,

$$\text{Ind}_{C_2}^{D_3} V_0 \simeq V_2 \oplus V_0.$$

Similarly,

$$\dim \text{Hom}_{D_3}(V_2, \text{Ind}_{C_2}^{D_3} V_s) = \dim \text{Hom}_{C_2}(\text{Res}_{C_2}^{D_3} V_2, V_s) = 1;$$

$$\dim \text{Hom}_{D_3}(V_0, \text{Ind}_{C_2}^{D_3} V_s) = \dim \text{Hom}_{C_2}(\text{Res}_{C_2}^{D_3} V_0, V_s) = 0;$$

$$\dim \text{Hom}_{D_3}(V_s, \text{Ind}_{C_2}^{D_3} V_s) = \dim \text{Hom}_{C_2}(\text{Res}_{C_2}^{D_3} V_s, V_s) = 1.$$

Therefore,

$$\text{Ind}_{C_2}^{D_3} V_s \simeq V_2 \oplus V_s.$$