November 19, 2024

Written assignment

Exercise 1. (6 pts)

(a) Let (V, ρ) be the complex irreducible 2-dimensional representation of $\mathbb{C}[D_3]$, given in the basis (e_1, e_2) by the matrices (here and below $\xi = e^{\frac{2\pi i}{3}}$):

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(r) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}$$

If (e_1, e_2) and (e_1', e_2') are bases in two copies of V, with the action of the generators in each copy given by the above matrices, is $e_1 \oplus e_1'$ a cyclic vector in $V \oplus V$? If not, find a cyclic vector in $V \oplus V$.

(b) Let G be a finite group, and V be a complex irreducible 3-dimensional representation of $\mathbb{C}[G]$. Which of the following representations of $\mathbb{C}[G]$ are cyclic?

$$V \oplus V, \qquad V \oplus V \oplus V, \qquad V \oplus V \oplus V \oplus V, \qquad V^{\oplus 6}.$$

In each case explain how to construct a cyclic vector, or why such vector does not exist.

(c) If $\{V_i\}$ is a finite set of (possibly repeating) complex irreducible representations of a semisimple finite dimensional algebra A, find the necessary and sufficient condition for the representation $V = \oplus_i V_i$ to be cyclic.

Solution 1. (a) We will consider the basis $\{u_1 = e_1 \oplus e_1', u_2 = e_1 \oplus e_2', u_3 = e_2 \oplus e_1', u_4 = e_2 \oplus e_2'\}$ in $V \oplus V$. Then

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We easily compute the action of the elements of D_3 in this basis:

$$\rho(r)(u_1) = \xi u_1, \quad \rho(r^2)(u_1) = \xi^{-1} u_1, \quad \rho(s)(u_1) = u_4, \quad \rho(rs)(u_1) = \xi^{-1} u_4, \quad \rho(r^2 s)(u_1) = \xi u_4.$$

This implies that $\mathbb{C}[D_3](u_1) = \mathrm{Span}_{\mathbb{C}}\{u_1, u_4\} \subset V \oplus V$, a 2-dimensional subspace. Therefore, $u_1 \in V \oplus V$ is not a cyclic vector.

On the other hand, let us take $u_2 = e_1 \oplus e_2' \in V \oplus V$. For convenience we introduce another basis:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice that $u_1 = v_1 + v_3$, $u_2 = v_1 + v_4$, $u_3 = v_2 + v_3$, $u_4 = v_2 + v_4$. We compute:

$$\rho(r)(u_2) = \xi v_1 + \xi^{-1} v_4, \quad \rho(r^2)(u_2) = \xi^{-1} v_1 + \xi v_4,$$

$$\rho(s)(u_2) = v_2 + v_3, \quad \rho(rs)(u_2) = \xi^{-1} v_2 + \xi v_3, \quad \rho(r^2 s)(u_2) = \xi v_2 + \xi^{-1} v_3.$$

These vectors clearly span $V \oplus V$ (in fact, it is sufficient to have $\mathrm{Span}_{\mathbb{C}}\{\rho(1)(u_2), \rho(r)(u_2), \rho(s)(u_2), \rho(rs)(u_2)\}$. Therefore, $e_1 \oplus e_2'$ is a cyclic vector and $V \oplus V$ is a cyclic representation of $\mathbb{C}[D_3]$.

(b) By Density theorem we know that $\rho : \mathbb{C}[G] \to \mathrm{End}(V)$ is surjective. This means that for any set of linearly independent vectors $\{v_1, v_2, \dots v_k\}$ in V and any set of vectors $\{u_1, u_2, \dots u_k\}$ there exist an element $a \in \mathbb{C}[G]$ such that $\rho(a) : \{v_1, v_2 \dots v_k\} \to \{u_1, u_2, \dots u_k\}$. Let V be an irreducible n-dimensional representation of $\mathbb{C}[G]$. Therefore a representation $V^{\oplus k}$ of $\mathbb{C}[G]$ is cyclic if and only if $k \le n = \dim(V)$ and a cyclic vector can be taken to be

$$v_1^{(1)} \oplus v_2^{(2)} \oplus \dots \oplus v_k^{(k)} \in V^{\oplus k},$$

where $v_i^{(i)}$ is in the i-th component of the direct sum $V^{\oplus k}$ and the vectors $\{v_i\}_{i=1}^k$ are linearly independent. Thus the representations $V \oplus V$ and $V \oplus V \oplus V$ are cyclic, and the remaining two are not.

(c) We can use the property of a cyclic representation of an associative algebra A proven in PS4, Ex. 2: a representation W of A is cyclic if and only if $W \simeq A/I$ for a proper left ideal $I \in A$. Recall that left ideals in A are equivalently subrepresentations of the left regular representation A_{reg} on itself. Since the algebra A is semisimple, it is isomorphic to a direct sum of matrix algebras $\mathrm{End}(V)$, each decomposing into a direct sum of $\dim(V)$ copies of V. This follows because V is the only irreducible representation of $\mathrm{End}(V)$ (see PS 4, Ex. 4). So we have for the set $\{V_i\}$ of inequivalent irreducible representations of A:

$$A_{reg} \simeq \bigoplus_{i=1}^{r} V_i^{\oplus \dim(V_i)}.$$

Any proper left ideal in $I \subset A$ is a subrepresentation of the left regular representation, and therefore a direct sum of several copies of irreducible representations of A with multiplicities of each irreducible representation smaller than its dimension. Notice that since A is semisimple, a quotient is also a subrepresentation. Therefore for a representation $V = \oplus V_i$ to be cyclic, it is necessary and sufficient for the multiplicity of each direct summand to be less of equal to its dimension.

Exercise 2. (9 pts) Let $F(G, \mathbb{C})$ be the space of functions on G and define the convolution product

$$f_1 \star f_2(z) = \sum_{xy=z \in G} f_1(x) f_2(y).$$

(a) Show that the convolution product in $F(G, \mathbb{C})$ is associative and find the neutral element.

(b) Show that the subspace of class functions $F_c(G, \mathbb{C}) \subset F(G, \mathbb{C})$ is a commutative subalgebra in $F(G, \mathbb{C})$.

(c) Let V and W be two irreducible representations of G. Compute $\chi_V \star \chi_W$. *Hint:* Write the characters in terms of the matrix elements of the representations and use the third orthogonality relation for the matrix elements of irreducible representations (see Lecture 7).

(d) Find the primitive idempotents in the convolution algebra $F_c(G, \mathbb{C})$: the solutions of the equation $f \star f = f$ such that f is not a sum of other nonzero solutions. Express the primitive idempotents in the basis of irreducible characters.

(e) Let $D_3 = \langle r, s \mid r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of order 6. Describe the irreducible complex representations of D_3 and compute its character table. Construct the set of primitive idempotents with respect to the convolution product in the space of the class functions on D_3.

Solution 2. (a) We can rewrite the convolution product as follows

$$f_1 \star f_2(z) = \sum_{x \in G} f_1(x) f_2(x^{-1}z).$$

To show that the convolution product is associative we compute

$$(f_1 \star f_2) \star f_3(z) = \sum_{x \in G} (f_1 \star f_2)(x) f_3(x^{-1}z) = \sum_{x \in G} \sum_{t \in G} f_1(t) f_2(t^{-1}x) f_3(x^{-1}z).$$

$$f_1 \star (f_2 \star f_3)(z) = \sum_{y \in G} f_1(y) (f_2 \star f_3)(y^{-1}z) = \sum_{y \in G} \sum_{s \in G} f_1(y) f_2(s) f_3(s^{-1}y^{-1}z).$$

With the change of variable $x = ts$ in the first line and $y = t$ in the second line, we deduce the equality. The neutral element of the convolution product is the characteristic function of the group unit $\delta_e(z) = \delta_{e,z}$. For any $f \in F(G, \mathbb{C})$ we have

$$f \star \delta_e(z) = \sum_{x \in G} f(x) \delta_e(x^{-1}z) = \sum_{x \in G} f(x) \delta_{x,z} = f(z) \quad \delta_e \star f(z) = \sum_{x \in G} \delta_e(x) f(x^{-1}z) = f(z).$$

(b) Let now f_1, f_2 be class functions. Then

$$f_1 \star f_2(z) = \sum_{x \in G} f_1(x) f_2(x^{-1}z) = \sum_{x \in G} f_1(z^{-1}xz) f_2(x^{-1}z) = \sum_{y \in G} f_2(y) f_1(y^{-1}z) = f_2 \star f_1.$$

Here we used the property of the class function $f_1(z^{-1}xz) = f_1(z)$ and the change of variable $y = x^{-1}z$.

(c) Let $n = \dim(V)$ and $m = \dim(W)$. Let us compute the convolution product of two irreducible characters:

$$\chi_V \star \chi_W(z) = \sum_{g\in G} \chi_V(x)\chi_W(x^{-1}z) = \sum_{g\in G} \chi_V(x) \sum_{i=1}^{m} t_{ii}^W(x^{-1}z),$$

where t_{ii}^W is the matrix element of the representation ρ_W. Using matrix multiplication and the definition of the dual representation, we have

$$t_{ii}^W(x^{-1}z) = \sum_{k=1}^{m} t_{ik}^W(x^{-1})t_{ki}^W(z) = \sum_{k=1}^{m} \overline{t_{ki}^W(x)}t_{ki}^W(z),$$

then we continue the computation

$$\chi_V \star \chi_W(z) = \sum_{x\in G}\sum_{j=1}^{n} t_{jj}^V(x) \sum_{i,k} \overline{t_{ki}^W(x)}t_{ki}^W(z) = \sum_{i,k=1}^{m}\sum_{j=1}^{n} t_{ki}^W(z) \sum_{x\in G} t_{jj}^V(x)\overline{t_{ki}^W(x)}.$$

Recall the third orthogonality relation:

$$\frac{1}{|G|}\sum_{g\in G} t_{ij}^V(g)\overline{t_{kl}^W(g)} = \frac{1}{\dim(V)}\delta_{V,W}\delta_{ik}\delta_{jl}.$$

Therefore, we have

$$\chi_V \star \chi_W(z) = \sum_{i,k=1}^{m}\sum_{j=1}^{n} t_{ki}^W(z)\frac{|G|}{\dim(V)}\delta_{V,W}\delta_{jk}\delta_{ji} = \frac{|G|}{\dim(V)}\delta_{V,W}\sum_{j=1}^{n} t_{jj}^V(z) = \frac{|G|}{\dim(V)}\delta_{V,W}\chi_V(z).$$

(d) From (c) we can easily find a set of orthogonal idempotents. Recall that the characters $\{\chi_V\}_{V\in Irr}$ form a basis in $F_c(G,\mathbb{C})$. We have proven that the characters are orthogonal with respect to the convolution product. We just need to introduce the correct normalization to get the idempotent property. We set

$$\psi_V = \frac{\dim(V)}{|G|}\chi_V.$$

Then $\{\psi_V\}_{V\in Irr}$ is the set of primitive idempotents with respect to the convolution product. Indeed,

$$\psi_V \star \psi_W(g) = \delta_{V,W}\frac{(\dim(V))^2}{|G|^2}\frac{|G|}{\dim(V)}\chi_V(g) = \delta_{V,W}\frac{\dim(V)}{|G|}\chi_V(g) = \delta_{V,W}\psi_V(g).$$

If $f \star f = f \in F_c(G,\mathbb{C})$, then we can decompose f in the basis of $\{\psi_V\}_{V\in Irr}$, $f = \sum_{V\in Irr} a_V\psi_V$. Then

$$f \star f = \sum_{V\in Irr} a_V^2\psi_V = f = \sum_{V\in Irr} a_V\psi_V,$$

which is possible if and only if $a_V \in \{0,1\}$. Therefore all other solutions of the equation $f \star f = f$ are sums of ψ_V's.

Remark: This implies that the characters of the irreducible representations can be defined independently of the representation theory, by finding the primitive idempotent of the convolution algebra of the class functions on the group and renormalizing them. Notice that we can express $\dim(V)$ in terms of the primitive idempotents:

$$\psi_V(e) = \frac{(\dim(V))^2}{|G|} \quad\Longrightarrow\quad \dim(V) = \sqrt{|G|\psi_V(e)}.$$

Then

$$\chi_V(g) = \sqrt{\frac{|G|}{\psi_V(e)}}\psi_V(g)$$

are the characters of the irreducible representations of G. This is a way to define characters independently of the representation theory.

(e) According to the argument in PS 7, Ex.1 there are three irreducible representations of D_3: the trivial 1-dimensional V_0, the 1-dimensional sign representation V_s and the 2-dimensional irreducible representation V_2 given by the symmetries of an equilateral triangle on a plane. Set $\xi = e^{2\pi i/3}$. We have

$$\rho_0(r) = \rho_0(s) = 1, \quad \rho_s(r) = 1, \quad \rho_s(s) = -1, \quad \rho_2(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_2(r) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

The conjugacy classes in D_3 are $\{(1), (r, r^2), (s, sr, sr^2)\}$. The character table is given by

	(1)	(r, r^2)	(s, sr, sr^2)
χ_{V_0}	1	1	1
χ_{V_s}	1	1	-1
χ_{V_2}	2	-1	0

The primitive idempotents in the convolution algebra are given by $\psi_V = \frac{\dim(V)}{|G|}\chi_V$:

	(1)	(r, r^2)	(s, sr, sr^2)
ψ_{V_0}	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
ψ_{V_s}	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
ψ_{V_2}	$\frac{2}{3}$	$-\frac{1}{3}$	0