December 2024

List of theorems

Below and at the exam all representations are supposed to be complex and finite dimensional.

Theorem 1. (PS 1) A representation of a finite group G is equivalent to a representation of the group algebra $\mathbb{C}[G]$.

Theorem 2. (Lecture 2) (Schur's lemma).
Let V_1, V_2 be representations of an algebra A. Let $\varphi : V_1 \to V_2$ be a nonzero homomorphism of representations. Then

(a) If V_1 is irreducible, then φ is injective.

(b) If V_1 is irreducible, then φ is surjective.

(c) If V_1 and V_2 are irreducible representations over $\mathbb{C}$, then $\varphi = \lambda\,\mathrm{Id}$, where $\lambda \in \mathbb{C}^*$.

Theorem 3. (Lecture 2). Every irreducible finite dimensional representation of a commutative algebra is one-dimensional.

Theorem 4. (Lecture 3) (Maschke's theorem).
Let G be a finite group. Then every finite dimensional complex representation of G is completely reducible, meaning that it is isomorphic to a direct sum of irreducible representations. The algebra $\mathbb{C}[G]$ is semisimple.

Theorem 5. (Lecture 3) (Weyl's unitary trick).

(a) Every finite dimensional complex representation of a finite group is unitary.

(b) Every finite dimensional unitary representation of any group is completely reducible.

Theorem 6. (Lecture 4). Let A be an associative algebra.

(a) If $I \subset A$ is a left ideal, then $I \subset A$ is a subrepresentation of the left regular representation, and A/I is a quotient representation.

(b) V is cyclic if and only if $V \simeq A/I$ for some left ideal $I \subset A$.

(c) V is irreducible if and only if every vector $v \in V$ is cyclic.

(d) V is irreducible if and only if $V \simeq A/I_{\max}$ for some maximal left ideal $I_{\max}$.

Theorem 7. (Lecture 4) (Density lemma)
Let V_i, $1 \le i \le m$ be irreducible finite dimensional complex representations of an algebra A. Let $V \simeq \oplus_{i=1}^n V_i^{\oplus n_i}$. Suppose that $W \subset V$ is a subrepresentation. Then $W \simeq \oplus_{i=1}^n V_i^{\oplus r_i}$, where $r_i \le n_i$ for all i, and the inclusion $\varphi_i : V_i^{\oplus r_i} \to V_i^{\oplus n_i}$ is given by a constant $r_i \times n_i$ matrix X_i with complex entries.

Theorem 8. (Lecture 4) (Density theorem)
Let (V, ρ) be an irreducible finite dimensional complex representation of A.

(a) Let $\{v_1, \dots v_n\} \subset V$ be linearly independent vectors, and let $\{w_1, \dots w_n\} \subset V$ be any set of vectors. Then there exist $a \in A$ such that $\rho(a)v_i = w_i$ for all $1 \le i \le n$.

(b) The map $\rho : A \to \mathrm{End}(V)$ is surjective.

Theorem 9. (PS 4) (Lecture 5) An irreducible representation of a matrix algebra $\mathrm{Mat}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$. Irreducible representations of a finite direct sum of matrix algebras $A = \oplus_{i\in I}\mathrm{Mat}_{n_i}(\mathbb{C})$ are $\{\mathbb{C}^{n_i}\}_{i\in I}$.

Theorem 10. (Lecture 5) (Structure theorem for finite dimensional algebras).
A finite dimensional complex algebra A has only finitely many inequivalent irreducible representations. Each irreducible representation is finite dimensional and

$$A/\mathrm{Rad}(A) \simeq \bigoplus_{i=1}^{n} \mathrm{End}(V_i),$$

where $\{V_i\}_{i=1}^n$ is a complete list of inequivalent irreducible representations of A.

Theorem 11. (Lecture 5) (Structure theorem for semisimple finite dimensional algebras).
Let A be a finite dimensional complex algebra. Then

$$A \text{ is semisimple} \iff A \simeq \bigoplus_{i=1}^{n} \mathrm{Mat}_{n_i}(\mathbb{C}).$$

Theorem 12. (Lecture 5) (Linear independence of characters).
Let A be a complex associative algebra.

(a) Characters of inequivalent irreducible finite dimensional representations of A are linearly independent.

(b) If A is a finite dimensional semisimple algebra, then the irreducible characters form a basis in $(A/[A,A])^*$.

Theorem 13. (Lecture 6) (Linear independence of characters of a finite group)
Let G be a finite group. The characters of complex irreducible representations of G form a basis in the space of complex-valued class functions on G.

Theorem 14. (Lecture 6). Let G be a finite group. The number of isomorphism classes of irreducible representations of G equals to the number of conjugacy classes of G. Any finite dimensional complex representation of G is determined by its character: $\chi_V = \chi_W \iff V \simeq W$.

Theorem 15. (Lecture 6). Let G be a finite group. Then $\mathbb{C}[G]$ is semisimple and

$$\mathbb{C}[G] \simeq \bigoplus_{i=1}^{r} \mathrm{End}(V_i),$$

where $\{V_i\}_{i=1}^r$ is the complete list of inequivalent irreducible representations of G, and r is the number of its conjugacy classes.

Theorem 16. (Lecture 6). Let V, W be finite dimensional complex representations of a finite group G. The character of the dual representation V^* is $\chi_{V^*}(g) = \overline{\chi_V(g)}$. The character of the tensor product $V \otimes W$ is $\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$.

Theorem 17. (Lecture 6) Let V, W be finite dimensional complex representations of a finite group G. Then

$$W \otimes V^* \simeq \mathrm{Hom}(V, W)$$

as a representation of G.

Theorem 18. (Lecture 7) (1st orthogonality relation).
For any finite dimensional complex representations V, W of G we have

$$(\chi_V, \chi_W) := \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \dim \mathrm{Hom}_G(V, W).$$

If V, W are irreducible, then

$$(\chi_V, \chi_W) = \begin{cases} 1, & V \simeq W \\ 0, & V \not\simeq W \end{cases}$$

Theorem 19. (Lecture 7) (2nd orthogonality relation).
Let $g, h \in G$ and $Z_g = \{t \in G : tgt^{-1} = g\}$ the centralizer subgroup of g. Then

$$\sum_{V \in \mathrm{Irr}(G)} \chi_V(g)\overline{\chi_V(h)} = \begin{cases} |Z_g| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{otherwise} \end{cases}$$

Theorem 20. (Lecture 7) (3rd orthogonality relation).

(a) Matrix elements of inequivalent irreducible representations of G are orthogonal under the form $(f_1, f_2) = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$.

(b)

$$(t_{ij}^V, t_{kl}^V) = \delta_{i,k}\delta_{j,l}\frac{1}{\dim(V)}.$$

Theorem 21. (Lecture 8) (Symmetric and exterior product of vector spaces).

(a) Let $\{e_i\}_{i=1}^k$ be a basis in V. Then S^nV has a basis $\{e_{i_1}e_{i_2}\dots e_{i_n}\}_{1\le i_1\le i_2\le\dots\le i_n\le k}$ and $\dim(S^nV)=\binom{n+k-1}{n}$.

(b) Let $\{e_i\}_{i=1}^k$ be a basis in V. Then $\wedge^nV$ has a basis $\{e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_n}\}_{1\le i_1<i_2<\dots<i_n\le k}$, where $n\le k$, and $\dim(\wedge^nV)=\binom{k}{n}$.

Theorem 22. (Lecture 8) (Action of G on $V^{\otimes n}$).
Let V be a finite dimensional complex representation of G. The action of G on $V^{\otimes n}$ commutes with the action of S_n by permutation of factors. Every S_n-isotypical component in $V^{\otimes n}$ is a subrepresentation of G in $V^{\otimes n}$.

Theorem 23. (Lecture 8) (Frobenius-Schur indicator).
Let V be an irreducible complex representation of a finite group G.

- If $V\not\simeq V^*$, then V is of complex type;
- If $V\simeq V^*$ and $V_0\subset S^2V$, then V is of real type;
- If $V\simeq V^*$ and $V_0\subset\wedge^2V$, then V is of quaternionic type.

Then the number of involutions $\{g\in G: g^2=1\}$ equals to $\sum_{V\text{ real}}\dim(V)-\sum_{V\text{ quaternionic}}\dim(V)$.

Theorem 24. (Lecture 9)
Let G be a finite group and V a complex irreducible representation of G. Then $\dim(V)$ divides $|G|$.

Theorem 25. (Lecture 9)
Let V be an irreducible complex representation of a finite group G, and C a conjugacy class in G such that $\gcd(|C|,\dim(V))=1$. Then either $\chi_V(g)=0\ \forall g\in C$, or any $g\in C$ acts as a scalar in V.

Theorem 26. (Lecture 9) (Burnside's theorem).
A group of order p^aq^b, where p,q are primes, is solvable.

Theorem 27. (Lecture 10) (Frobenius formula for the character of an induced representation).
Let $H\subset G$ be a subgroup and $\{x_\sigma\}_{\sigma\in H\backslash G}$ representatives of right cosets of H in G. Then the character χ of Ind_H^GV is given by

$$\chi(g)=\sum_{\sigma\in H\backslash G\ :x_\sigma g x_\sigma^{-1}\in H}\chi_V(x_\sigma g x_\sigma^{-1})=\frac{1}{|H|}\sum_{x\in G\ :xgx^{-1}\in H}\chi_V(xgx^{-1}).$$

Theorem 28. (Lecture 10) (Frobenius reciprocity).
Let $H\subset G$ be a subgroup, V a representation of G, W a representation of H. Then

$$\mathrm{Hom}_G(V,\ \mathrm{Ind}_H^GW)\simeq\mathrm{Hom}_H(\mathrm{Res}_H^GV,\ W).$$

Theorem 29. (Lecture 11) (Induction and restriction).
Let $H\subset G$ be a subgroup, V a representation of G, W a representation of H. Then

$$\mathrm{Ind}_H^GW\simeq\mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G],W)\simeq\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W.$$

$$\mathrm{Res}_H^GV\simeq\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G],V)\simeq\mathbb{C}[G]\otimes_{\mathbb{C}[G]}V.$$

Theorem 30. (PS 13). (Transitivity of the induction) Let $K\subset H\subset G$ be subgroups of a finite group G and V a complex representation of K. Then

$$\mathrm{Ind}_H^G\,\mathrm{Ind}_K^HV\simeq\mathrm{Ind}_K^GV.$$

Theorem 31. (Lecture 11) (Specht module for S_n).
Let $\lambda=(\lambda_1\ge\dots\ge\lambda_p)$ be a partition of n and Y_λ a Young diagram with rows λ_i filled with numbers from 1 to n. Let P_λ be the subgroup of permutations along the columns and Q_λ the subgroup of permutations along the rows of Y_λ. Let $a_\lambda=\sum_{g\in P_\lambda}g$, $b_\lambda=\sum_{g\in Q_\lambda}(-1)^gg$, and $c_\lambda=a_\lambda b_\lambda$. Then

$$V_\lambda=\mathbb{C}[S_n]c_\lambda$$

is the Specht module corresponding to the partition λ. The Specht modules $\{V_\lambda\}_{\lambda\text{ partition of }n}$ form the complete set of inequivalent irreducible representations of S_n.

Theorem 32. (Lecture 12).
Let A be an associative algebra and M a left A-module. Let $e \in A$ be an idempotent: $e^2 = e$. Then

$$\mathrm{Hom}_A(Ae, M) \simeq eM.$$

Theorem 33. (Lecture 12). (Induced representations of S_n)
Let λ be a partition of n and define

$$U_\lambda = \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{\mathrm{triv}} \simeq \mathbb{C}[S_n] a_\lambda.$$

Then $U_\lambda = \oplus_{\mu \geq \lambda} K_{\lambda,\mu} V_\mu$, where V_μ are the Specht modules and $K_{\lambda,\mu}$ the Kostka numbers, $K_{\lambda,\lambda} = 1$.

Theorem 34. (Lecture 12) (Character of U_λ).
Let $C_{\mathbf{i}}$ be the conjugacy class in S_n of cycle type $(i_1, i_2, \ldots i_l, \ldots)$ where i_l is the number of cycles of length l. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p)$ be a partition of n. Let $N \geq p$ and $\{x_1, \ldots x_N\}$ be variables. Then the character $\chi_{U_\lambda}(C_{\mathbf{i}})$ is equal to the coefficient of $x^\lambda = \prod_j x_j^{\lambda_j}$ in the polynomial

$$\prod_{m \geq 1} H_m(x)^{i_m}$$

where $H_m(x) = (x_1^m + x_2^m + \ldots + x_N^m)$.

Theorem 35. (Lecture 12) (Character of the Specht module V_λ).
Let $C_{\mathbf{i}}$ be the conjugacy class in S_n of cycle type $(i_1, i_2, \ldots i_l, \ldots)$ where i_l is the number of cycles of length l. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p)$ be a partition of n. Let $N \geq p$ and $\{x_1, \ldots x_N\}$ be variables. Then the character $\chi_{U_\lambda}(C_{\mathbf{i}})$ is equal to the coefficient of $x^{\lambda+\rho} = \prod_j x_j^{\lambda_j + N - j}$ in the polynomial

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{i_m},$$

where

$$\Delta(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j), \qquad H_m(x) = (x_1^m + x_2^m + \ldots + x_N^m).$$

Theorem 36. (Lecture 13) (Hook length formula).
Let V_λ be the Specht module corresponding to the partition λ of n. Then

$$\mathrm{dim} V_\lambda = \frac{n!}{\prod_{i \in \lambda_j} h(i,j)},$$

where $h(i,j)$ is the number of squares in the hook to the right and down from the square (i,j) in Y_λ.

Theorem 37. (PS 13) (Induction from S_{n-1} to S_n)
Let V_λ denote the Specht module for S_n, where λ is a partition of n. Then

(a) $\mathrm{Res}_{S_{n-1}}^{S_n} V_\lambda \simeq \bigoplus_{\mu \in R(\lambda)} V_\mu$, where $R(\lambda)$ is the set of Young diagrams obtained by removing one square from Y_λ.

(b) $\mathrm{Ind}_{S_{n-1}}^{S_n} V_\mu \simeq \bigoplus_{\lambda \in A(\mu)} V_\lambda$, where $A(\mu)$ is the set of Young diagrams obtained by adding one square from Y_μ.

Theorem 38. (Lecture 14) (Double centralizer property).
Let E be a finite dimensional complex vector space and A, B two subalgebras of $\mathrm{End}(E)$. Suppose that A is semisimple and $B = \mathrm{End}_A(E)$. Then

(a) $A = \mathrm{End}_B(E)$ (the centralizer of the centralizer of A is A).

(b) B is semisimple.

(c) $E = \bigoplus_{i \in I} V_i \otimes W_i$ as a representation of $A \otimes B$, where $\{V_i\}_{i \in I}$ are all the irreducible representations of A and $\{W_i\}_{i \in I}$ are all the irreducible representations of B.

Theorem 39. (Lecture 14) (Schur-Weyl duality).
Let V be a finite dimensional complex vector space and $GL(V)$ the group of invertible linear maps in V. Then we have

$$V^{\otimes n} \simeq \bigoplus_\lambda V_\lambda \otimes L_\lambda,$$

where λ runs over the partitions of n, V_λ are the Specht modules for S_n, and $L_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ are distinct irreducible representations of $GL(V)$, or zero.