Lecture 8.

Recall: Three orthogonality relations:

Thm 1 If V, W are irreducible representations of G $\Rightarrow$

$$(\chi_V, \chi_W) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \begin{cases} 1, & \text{if } V \simeq W \\ 0, & \text{otherwise} \end{cases}$$

Thm 2. If $g, h \in G$ and $Z_g = \{t \in G : tgt^{-1} = g\}$ the centralizer $\Rightarrow$

$$\sum_{V\in \operatorname{Irr} G} \chi_V(g)\overline{\chi_V(h)} = \begin{cases} |Z_g|, & \text{if } h = kgk^{-1} \text{ for some } k \in G \\ 0 & \text{otherwise.} \end{cases}$$

Thm 3. Let $V \in \operatorname{Irr} G$, $\{v_i\}_{i=1}^n$ orthonormal basis wrt a G-invariant Hermitian form on V. Let $t_{ij}^V(x) = \langle \rho_V(x) v_i, v_j \rangle$ $\forall x \in G$. $\Rightarrow$

$$(t_{ij}^V, t_{kl}^W) = \frac{1}{|G|}\sum_{g\in G} t_{ij}^V(g)\overline{t_{kl}^W(g)} = \begin{cases} \delta_{ik}\delta_{jl}\frac{1}{\dim V}, & \text{if } V \simeq W \\ 0, & \text{otherwise.} \end{cases}$$

Today: tensor powers of representations of G.

Example of a character table.

Let $G = A_4$. Conjugacy classes: $\{ \overset{1}{1}, \overset{3}{\{(12)(34)\}}, \overset{4}{\{(123)\}}, \overset{4}{\{(132)\}} \}$

Find a normal subgroup. $K = \{1, (12)(34), (13)(24), (14)(32)\}$ the Klein group

$$K \lhd A_4 \implies A_4/K \simeq C_3$$

Suppose (U, ρ) is an irreducible representation of $G/K = H$. Then

$\varphi : G \to H$ group homomorphism

Let $\tilde{\rho}(g) = \rho \circ \varphi(g)$ is a representation of G; it is also irreducible.

If $W \subset U$ a subrepresentation of $G \implies$ it is also a subrepresentation in H.

Irreducible representations of $\underset{\langle t : t^3 = 1 \rangle}{C_3 \,\shortparallel}$? $\implies$

V_0 trivial $\quad \rho(t) = 1$

$V_\zeta \quad \rho(t) = \zeta, \quad \zeta = e^{\frac{2\pi i}{3}}$

$V_{\zeta^2} \quad \rho(t) = \zeta^2$

	1	$\overset{3}{(12)(34)}$	$\overset{4}{(123)}$	$\overset{4}{(132)}$	
V_0	1	1	1	1	real type $FS(V_0)=1$
V_ζ	1	1	ζ	ζ^2	complex type $FS(V_\zeta)=0$
V_{ζ^2}	1	1	ζ^2	ζ	complex type $FS(V_{\zeta^2})=0$.
V_3	3	-1	0	0	real type $FS(V_3)=1$

$$|A_4| = 12 = \sum_{i=1}^{4} (\dim V_i)^2 = 1+1+1+(\dim V_3)^2 \Rightarrow \dim V_3 = 3$$

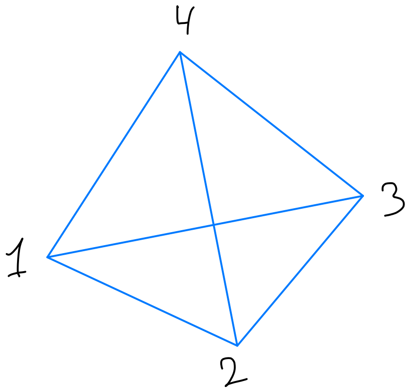
$$\frac{1}{12}(3+3a+4b+4c)=0 \quad \Rightarrow \quad 4b+4c=0 \Rightarrow b=-c$$

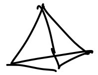
$$\frac{1}{12}(3+3a+4b\bar{\zeta}+4c\bar{\zeta}^2)=0 \quad \Rightarrow \quad b(\bar{\zeta}-\bar{\zeta}^2)=0 \Rightarrow b=c=0$$

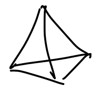
$$\frac{1}{12}(3+3a+4b\bar{\zeta}^2+4c\bar{\zeta})=0$$

$$3+3a=0 \Rightarrow a=-1$$

V_3



S_4 = isometries of 

A_4 = rotations of  $\longleftrightarrow V_3$

$$r_\pi = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad \operatorname{tr} r_\pi = -1 = \chi_{V_3}((12)(34))$$

$$r_{\frac{2\pi}{3}} = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \zeta & \\ 0 & & \zeta^2 \end{pmatrix} \qquad \operatorname{tr} r_{\frac{2\pi}{3}} = \chi((123)) = 0$$

Similarly for $\chi((132))$.

$$V_\zeta \otimes V_\zeta \simeq V_{\zeta^2} \quad ; \quad V_{\zeta^2} \otimes V_{\zeta^2} \simeq V_\zeta$$

$V_3 \otimes V_3$:

		3		
	1	(12)(34)	(123)	(132)
$V_3^{\otimes 2}$	9	1	0	0

$$\langle V_3^{\otimes 2}, V_0 \rangle = \frac{1}{12}(9+3) = 1$$

$$\langle V_3^{\otimes 2}, V_\zeta \rangle = \frac{1}{12}(9+3) = 1$$

$$\langle V_3^{\otimes 2}, V_{\zeta^2} \rangle = \frac{1}{12}(9+3) = 1$$

$$\langle V_3^{\otimes 2}, V_3 \rangle = \frac{1}{12}(27-3) = 2$$

$$\Rightarrow V_3 \otimes V_3 \simeq V_3^{\oplus 2} + V_0 + V_\zeta + V_{\zeta^2}$$

Tensor powers of representations.

Recall: V, W are G-representations $\Rightarrow$ $V \otimes W$ is a representation

via $\rho(g)\, v \otimes w = \rho_V(g) v \otimes \rho_W(g) w.$

$\Rightarrow$ can consider $V^{\otimes n} = V \otimes \dots \otimes V$

Def. Let V be a fin dim vector space. The n-th symmetric power

$$S^n V = V^{\otimes n} / \operatorname{Span}(T - \sigma(T)) \qquad \begin{array}{l} T = v_1 \otimes v_2 \otimes \dots \otimes v_n \\ \sigma(T) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)} \end{array}$$

Proposition Let $\{e_i\}_{i=1}^{k}$ be a basis in V, $\dim V = k$.

Then $S^n V$ has a basis $\{ e_{i_1} e_{i_2} \dots e_{i_n} \,,\; 1 \le i_1 \le \dots \le i_n \le k \}$

and $\dim S^n V = \binom{n+k-1}{n}$

Proof: $\sigma(T) = e_{\sigma(j_1)} \otimes \dots \otimes e_{\sigma(j_n)} = e_{j_1} \otimes \dots \otimes e_{j_n}$ $\Rightarrow$ you can reorder the components in the order : $1 \le j_1 \le j_2 \dots \le j_n \le k$

These elts are in bijection with $(n+k-1)$ dots

$n=3,\ k=4$

$$e_1 \quad e_1 \quad \Big| \quad e_2 \quad \Big| \quad e_3 \quad \Big| \quad e_4 \quad \to \quad e_1 e_1 e_2 e_3 e_4$$

$$\Rightarrow \quad \dim S^n V = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Ex. $\dim V = 3,\quad n = 2 \qquad \{e_1, e_2, e_3\}$ basis in V

$\Rightarrow \quad S^2 V$ has a basis $\leftrightarrow \{e_1e_1,\ e_1e_2,\ e_1e_3,\ e_2e_2,\ e_2e_3,\ e_3e_3\}$

$$\binom{2+3-1}{2} = \binom{4}{2} = \frac{4 \cdot 3}{2} = 6$$

$\dim V = 2,\quad n = 3 \qquad \{e_1, e_2\}$ basis in V

$S^3 V$ has a basis $\leftrightarrow \{e_1e_1e_1,\ e_1e_1e_2,\ e_1e_2e_2,\ e_2e_2e_2\}$

$$\binom{2+3-1}{3} = \binom{4}{3} = 4.$$

Def. Let V be a vector space, $\dim V = k$. The n-th exterior power of V

$$\Lambda^n V = V^{\otimes n} / \mathrm{Span}(T + s(T)) \quad , \; T = v_1 \otimes \ldots \otimes v_n$$

$s \in S_n$ a transposition

Proposition $\Lambda^n V$ has a basis $\leftrightarrow \{ e_{i_1} \wedge \ldots \wedge e_{i_n} \; , \; 1 \leq i_1 < i_2 < \ldots < i_n \leq k \}$
where $n \leq k$, $\dim \Lambda^n V = \binom{k}{n}$

Ex. $k = 3$, $n = 2 \Rightarrow \Lambda^2 V \leftrightarrow \{ e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 \}$

$V \leftrightarrow \{ e_1, e_2, e_3 \}$ $\qquad \dim \Lambda^2 V = 3 = \binom{3}{2}$

Remark. $V^{\otimes 2} \simeq S^2 V \oplus \Lambda^2 V \qquad \dim V = k \geq 2$

$$S^2 V \leftrightarrow \{ e_{i_1} e_{i_2} \; , \; 1 \leq i_1 \leq i_2 \leq k \}$$

$$\Lambda^2 V \leftrightarrow \{ e_{i_1} \wedge e_{i_2} \; , \; 1 \leq i_1 < i_2 \leq k \}$$

$\Rightarrow V^{\otimes 2} : \; e_i \otimes e_j \longrightarrow \left\{ \begin{array}{ll} e_i e_j \in S^2 V & \text{if } i \leq j \\ e_j \wedge e_i \in \Lambda^2 V & \text{if } i > j \end{array} \right\}$ gives the entire basis in $V^{\otimes 2}$

$$\dim S^2V + \dim \Lambda^2 V = \binom{k+1}{2} + \binom{k}{2} = \frac{k(k+1)}{2} + \frac{k(k-1)}{2} = k^2 = \dim V^{\otimes 2}$$

Tensor algebra.

Def. $TV \overset{def}{=} \bigoplus_{n\geq 0} V^{\otimes n}$, $V^0 = k$ and the multiplication is given by concatenation

$$a \cdot b = a \otimes b \quad \Rightarrow \quad V^{\otimes n} \cdot V^{\otimes m} \subset V^{\otimes (n+m)} \quad \text{graded}$$

If $\{x_1 \ldots x_k\}$ is a basis in $V \Rightarrow TV \simeq k\langle x_1 \ldots x_n\rangle$ free algebra

Def. Symmetric algebra $SV \simeq TV \big/ \langle w\otimes v - v\otimes w,\ \forall v, w \in V\rangle$

$$S^0V \simeq k, \quad S^1 V \simeq V$$

If $\{x_1 \ldots x_k\}$ is a basis in $V \Rightarrow SV \simeq k[x_1 \ldots x_k]$ the polynomial algebra

$$S^n V \cdot S^m V \subset S^{n+m} V$$

Def. Exterior algebra $\Lambda V \simeq TV / \langle v\otimes v,\ \forall v\in V\rangle$

equiv. $\langle v\otimes w + w\otimes v,\ \forall w, v\in V\rangle$

$$\dim \Lambda V = \sum_{i=0}^{k} \dim \Lambda^k V = \sum_{i=0}^{k} \binom{k}{i} = 2^k.$$

Symmetric group action on $V^{\otimes n}$

Def. Let V be a fin dim representation of G, $\rho: G \to GL(V)$

Then S_n acts on $V^{\otimes n}$ by permutation of factors

$$\varphi(\sigma)(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(n)}$$

Ex. Let $\{e_1, e_2, e_3, \ldots\}$ elements of a basis of V.

$$\varphi((123))((e_i+e_j)\otimes e_j \otimes e_k) = e_k \otimes (e_i+e_j)\otimes e_j$$

This defines a representation of S^n on $V^{\otimes n}$

$$\varphi(\sigma\mu)(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = u_{\sigma\mu(1)} \otimes u_{\sigma\mu(2)} \ldots \otimes u_{\sigma\mu(n)}$$

$$\varphi(\sigma)(u_{\mu(1)} \otimes u_{\mu(2)} \otimes \ldots u_{\mu(n)}) = \varphi(\sigma)\varphi(\mu)(u_1 \otimes u_2 \otimes \ldots \otimes u_n)$$

Proposition. The action of G on $V^{\otimes n}$ commutes with the action of S^n.

Proof:

$$\rho(g)\varphi(\sigma)(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = \rho(g)(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(n)})$$

$$= (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(n)})$$

where $\rho(g)\, u_j \overset{\text{def}}{=} v_j$

$$\varphi(\sigma)\rho(g)(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = \varphi(\sigma)(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = (v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})$$

▨

Corollary. Every S_n-isotypical component in $V^{\otimes n}$ is a subrepresentation of G in $V^{\otimes n}$.

Def. If $W = \bigoplus_k L_k^{\oplus n_k}$, where $\{L_k\}$ are inequivalent irreducible repres. of H $L_k^{\oplus n_k}$ is called *an isotypical component* of W with respect to the H-action.

Proof: By Schur's lemma any S_n-intertwiner map $V^{\otimes n} \to V^{\otimes n}$ preserves the isotypical components. (no nonzero S_n-invariant map $L_k \to L_m$ $k \neq m$)

We have $\rho(g): V^{\otimes n} \to V^{\otimes n}$ an S_n-intertwiner

$\Rightarrow L_k^{\oplus n_k}$ is a subrepresentation of G. ▨

In particular, $S^n V \subset V^{\otimes n}$ the trivial component [PS 9]

$\Lambda^n V \subset V^{\otimes n}$ the sign component

Frobenius-Schur indicator.

Def. Let V be a irreducible complex representation of a finite G.

(1) If $V \not\simeq V^*$ $\overset{def}{\Rightarrow}$ V is called of complex type $\quad FS(V) \overset{def}{=} 0$

(2) If $V \simeq V^* \Leftrightarrow V_0 < V \otimes V = S^2 V \oplus \Lambda^2 V$ as G-representations.

If $V_0 < S^2 V$ $\overset{def}{\Rightarrow}$ V is of real type $\quad FS(V) \overset{def}{=} 1$

(3) $V \simeq V^*$, and $V_0 < \Lambda^2 V$ $\overset{def}{\Rightarrow}$ V is of quaternionic type. $\quad FS(V) \overset{def}{=} -1$.

Lemma Let V be a fin dim representation of G.

Then $$\chi_V(g^2) = \chi_{S^2 V}(g) - \chi_{\Lambda^2 V}(g)$$ [PS8]

Proposition. $$\left|\{g\in G : g^2=1\}\right| = \sum_{V\in \mathrm{Irr}\,G} FS(V)\cdot \dim V =$$
$$= \sum_{V\ \mathrm{real}} \dim V - \sum_{V\ \mathrm{quat.}} \dim V.$$

Proof. $g^2\cdot \mathbb{C}[G]$: $g^2 h = h \iff g^2=1 \quad \forall h\in G$

$$tr|_{\mathbb{C}[G]}(g^2) = \begin{cases} 0, & \text{if } g^2\neq 1 \\ |G|, & \text{if } g^2=1 \end{cases}$$

$\mathbb{C}[G]\simeq \bigoplus V_i^{\oplus \dim V_i}$
as the left regular representation.

$$\Rightarrow \sum_{g\in G} tr|_{\mathbb{C}[G]}(g^2) = |G|\cdot \#(\text{involutions in } G)$$

$$\left|\{g\in G : g^2=1\}\right| = \frac{1}{|G|}\sum_{g\in G} tr_{\mathbb{C}[G]}(g^2) = \frac{1}{|G|}\sum_{g\in G}\sum_{V\in \mathrm{Irr}} \dim V\, \chi_V(g^2) =$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{V\in \mathrm{Irr}} \dim V\left(\chi_{S^2V}(g)-\chi_{\Lambda^2 V}(g)\right) = \sum_{V\in \mathrm{Irr}} \dim V\, \chi_{S^2V}\Bigg(\underbrace{\frac{1}{|G|}\sum_{g\in G} g}_{}\Bigg) -$$

P - projector onto the trivial component

(see lecture 7).

$$-\sum_{V\in Irr} \dim V\, \chi_{\Lambda^2 V}\Big(\underbrace{\frac{1}{|G|}\sum_{g\in G} g}_{=P}\Big) = \sum_{V\in Irr} \dim V \dim (S^2 V)^G - \sum_{V\in Irr} \dim V\,(\Lambda^2 V)^G =$$

by def $(S^2V)^G = 1$ if V of real type

$(\Lambda^2 V)^G = 1$ if V of quaternionic type

$$= \sum_{V \text{ real}} \dim V - \sum_{V \text{ quaternionic}} \dim V \quad = \#\text{ involutions in } G.$$

$\blacksquare$