Lecture 7.

Today: three orthogonality relations.

Written assignment: 19-26 november

Recall: First orthogonality relation:

Let $(f_1, f_2) = \frac{1}{|G|} \sum_{g\in G} f_1(g) \overline{f_2(g)}$ inner product in $F(G,\mathbb{C})$.

Theorem. For any finite dimensional representations V, W of G

we have $$(\chi_V, \chi_W) = \frac{1}{|G|} \sum_{g\in G} \chi_V(g) \overline{\chi_W(g)} = \dim \mathrm{Hom}_G(V,W)$$

In particular, if V and W are irreducible $\Rightarrow$

$$(\chi_V, \chi_W) = \begin{cases} 1, & V \simeq W \\ 0, & V \not\simeq W \end{cases}$$

Proof $(\chi_V, \chi_W) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_{W^*}(g) =$

$= \frac{1}{|G|}\sum_{g\in G}\chi_{V\otimes W^*}(g) = \chi_{V\otimes W^*}\left(\frac{1}{|G|}\sum_{g\in G} g\right) = \chi_{V\otimes W^*}(P)$

$P = \frac{1}{|G|}\sum_{g\in G} g$. Properties of P:

(1) $P \in Z(\mathbb{C}[G])$: $hP = \frac{1}{|G|}\sum_{g\in G} hg = Ph = \frac{1}{|G|}\sum_{g\in G} gh = P$

$PV \subset V$ a subrepresentation : $hPV = PV \quad \forall h \in G$

$P : V \to PV$ is an intertwiner : $hPV = PhV = PV$

(2) $PV \simeq$ trivial isotypical component : $hPv = Pv \quad \forall v \in V$

(3) $P^2 = P$: $P^2 = \frac{1}{|G|}\sum_{g\in G} g \cdot \frac{1}{|G|}\sum_{g\in G} g = \frac{1}{|G|^2}|G|\sum_{g\in G} g = \frac{1}{|G|}\sum_{g\in G} g = P$

$\Rightarrow$ P is a projector $\Rightarrow$ eigenvalues $0, 1$

(4) If $v \in \underset{\substack{\uparrow \\ \text{trivial}}}{V_0} \subset V$, $\Rightarrow$ $Pv = \frac{1}{|G|}\sum_{g\in G} gv = \frac{1}{|G|}|G|v = v$

$$(\chi_V, \chi_W) = \mathrm{Tr}_{V\otimes W^*}(P) = \dim \mathrm{Hom}(V_0, V\otimes W^*) = \dim (V\otimes W^*)^G =$$

Recall: $V\otimes W^* \overset{G\text{-repres.}}{\simeq} \mathrm{Hom}(W,V)$

$$(V\otimes W^*)^G \simeq \mathrm{Hom}_G(W,V)$$

$= \dim \mathrm{Hom}_G(W,V)$

$\Rightarrow$ In particular, if both V, W are irreducibles $\Rightarrow$

$$(\chi_V, \chi_W) = \begin{cases} 1, & \text{if } V \simeq W \\ 0, & \text{otherwise.} \end{cases}$$

$\blacksquare$

Example. $G = D_4$, character table.

$$(\chi_2, \chi_3) = \frac{1}{8}\big(1\cdot 1 + 2\cdot(-1)\cdot(-1) + 1\cdot 1 + 2\cdot(-1)\cdot 1 + 2\cdot(1)\cdot(-1)\big) = 0$$

$$(\chi_4, \chi_3) = \frac{1}{8}\big(1\cdot 2 + 1\cdot(-2)\big) = 0$$

$$(\chi_4, \chi_4) = \frac{1}{8}\big(2\cdot 2 + (-2)\cdot(-2)\big) = 1$$

	1	2	1	2	2
	1	r	r^2	s	sr
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	-1	1	-1	1
χ_3	1	-1	1	1	-1
2-dim χ_4	2	0	-2	0	0

Corollary. V is irreducible as a representation of G $\Longleftrightarrow$

$$(\chi_V, \chi_V) = 1$$

Proof: $V = \bigoplus V_i^{\oplus n_i}$ where $\{V_i\}$ are inequivalent irreducible representations.

$$\chi_V = \sum n_i \chi_{V_i} \Rightarrow \langle \chi_V, \chi_V\rangle = \sum n_i^2 \underbrace{\langle \chi_{V_i}, \chi_{V_i}\rangle}_{1} = \sum n_i^2 = 1$$

$$\langle \chi_{V_i}, \chi_{V_j}\rangle = 0, \quad i \neq j$$

$\Rightarrow \exists$ unique $n_i = 1 \Rightarrow V \simeq V_i$. ∎

Second orthogonality relation.

Theorem. Let $g, h \in G$ and let $Z_g = \{t \in G : tgt^{-1} = g\}$
centralizer subgroup of $g \in G$.

Then $\sum\limits_{V \in \operatorname{Irr} G} \chi_V(g) \cdot \overline{\chi_V(h)} = \begin{cases} |Z_g| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{otherwise} \end{cases}$

inequivalent irreducibles

Proof. We have $\overline{\chi_V(h)} = \chi_{V^*}(h) \Rightarrow$

$$\sum_{V \in \operatorname{Irr} G} \chi_V(g)\chi_{V^*}(h) = \operatorname{Tr}\Big|_{\underbrace{\bigoplus_{V \in \operatorname{Irr} G} V \otimes V^*}} \left(\rho_V(g) \otimes \rho_{V^*}(h)\right)$$

$$\bigoplus_{V \in \operatorname{Irr} G} V \otimes V^* \simeq \bigoplus_{V \in \operatorname{Irr} G} \operatorname{End} V \simeq \mathbb{C}[G]$$

Let $v \otimes w^* \in V \otimes V^*$; $v \otimes w^* : u \to v\, w^*(u)$; $\rho_V(g) \otimes \rho_{V^*}(h)(v \otimes w^*) =$

$= gv \otimes w^*(h^{-1}) \Rightarrow u \to g v w^*(h^{-1}u)$

$$\Rightarrow \begin{tikzcd} v\otimes w^* \arrow[r, "\rho_V(g)\otimes\rho_{V^*}(h)"] \arrow[d] & gv\otimes w^* h^{-1} \arrow[d] \\ x \arrow[r, "\rho_V(g)\otimes\rho_{V^*}(h)"] & g x h^{-1} \end{tikzcd} \Rightarrow$$

$$\sum_{V\in \mathrm{Irr}\, G} \chi_V(g)\chi_{V^*}(h) = \mathrm{Tr}\Big|_{\mathbb{C}[G]} \overset{F}{(x \to g x h^{-1})}$$

$F: \mathbb{C}[G] \to \mathbb{C}[G]$ given by a matrix in the basis of group elts

$\mathrm{Tr}_{\mathbb{C}[G]} F = \#$ of nonzero F_{ii} : $F_{ii} = 1 \Leftrightarrow g x h^{-1} = x$

$\Leftrightarrow gx = xh \Leftrightarrow g = x h x^{-1}$

$\Leftrightarrow \mathrm{Tr}\Big|_{\mathbb{C}[G]} (x \to g x h^{-1}) = 0$ if g not conjugate to h.

If $\exists x \in G: g = x h x^{-1}$, how many different x's ? : $x \to g x h^{-1} = x$

Suppose $g = yhy^{-1} \Rightarrow xhx^{-1} = yhy^{-1} \Rightarrow y^{-1}xhx^{-1}y = h = (y^{-1}x)h(y^{-1}x)^{-1}$

$\Rightarrow y^{-1}x = t \in Z_h \Rightarrow y = xt^{-1},\ t \in Z_h$, $|Z_h| = |Z_g|$

$\Rightarrow Tr\big|_{\mathbb{C}[G]} (x \to gxh^{-1}) = |Z_g|$, if g is conjugate to h.

$\blacksquare$

Conclusion

1^{st}: $$\frac{1}{|G|}\sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|}\sum_{C_g} |C_g|\, \chi_V(g)\overline{\chi_W(g)} =$$

$$= \sum_{C_g} \frac{1}{|Z_g|}\chi_V(g)\overline{\chi_W(g)} = \sum_{C_g} \frac{\chi_V(g)}{\sqrt{|Z_g|}}\,\overline{\frac{\chi_W(g)}{\sqrt{|Z_g|}}} = \begin{cases} 1, & V \simeq W \text{ irreducibles} \\ 0, & \text{otherwise} \end{cases}$$

2^{nd}: $$\sum_{V\in \mathrm{Irr}G} \chi_V(g)\overline{\chi_V(h)} = \sum_{V\in \mathrm{Irr}G} \frac{\chi_V(g)}{\sqrt{|Z_g|}} \cdot \overline{\frac{\chi_V(h)}{\sqrt{|Z_g|}}} = \begin{cases} 1, & h \text{ conj to } g \\ 0, & \text{otherwise.} \end{cases}$$

Example. $G = D_4$

	1	2	1	2	2
	1	r	r^2	s	sr
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	-1	1	-1	1
χ_3	1	-1	1	1	-1
χ_4	2	0	-2	0	0
$\lvert Z_g\rvert$	8	4	8	4	4

	1	2	1	2	2
	C_1	C_r	C_{r^2}	C_s	C_{sr}
$\frac{\chi_0}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\chi_1}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\chi_2}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{\chi_3}{\sqrt{\lvert Z_g\rvert}}$	$\frac{1}{\sqrt{8}}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{\chi_4}{\sqrt{\lvert Z_g\rvert}}$	$\frac{2}{\sqrt{8}}$	0	$-\frac{2}{\sqrt{8}}$	0	0

orthonormal columns
and rows

Third orthogonality relation.

Def. Let V be an irreducible representation of G, and $\{v_1 \dots v_n\}$ an orthonormal basis in V with respect to a G-invariant Hermitian inner product.

(Recall $\langle u,v\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \rho_V(g)u, \rho_V(g)v\rangle_0$ is G-invariant for any Hermitian inner product $\langle , \rangle_0$).

The matrix elements of ρ in V are $t_{ij}^V(x) = \langle \rho_V(x)v_i, v_j\rangle$.

Theorem. (1) Matrix elements of nonisomorphic irreducible representations are orthogonal in $F(G,\mathbb{C})$ under the form

$$(f_1, f_2) = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$$

(2) $\quad (t_{ij}^V, t_{ke}^V) = \delta_{ik}\delta_{je}\cdot\frac{1}{\dim V}$

<u>Proof.</u> Let $\{v_i\}$ and $\{w_i\}$ orthonormal bases in V and W irreducible repres.

Let $w_i^* \in W^*$ given by $w_i^*(u) = \langle u, w_i\rangle$

Matrix elts of $\rho(x)$ in W^* are complex conj to the matrix elts of $\rho(x)$ in W:

if $\; x w_i = \alpha_1 w_1 + \alpha_2 w_2 \quad \Rightarrow \quad x w_i^*(u) = \langle x^{-1}u, w_i\rangle \overset{G\text{-inv}}{=} \langle u, x w_i\rangle =$

$= \langle u, \alpha_1 w_1 + \alpha_2 w_2\rangle = \bar{\alpha}_1\langle u, w_1\rangle + \bar{\alpha}_2\langle u, w_2\rangle = \bar{\alpha}_1 w_1^*(u) + \bar{\alpha}_2 w_2^*(u).$

$$(t_{ij}^V, t_{ke}^W) = \frac{1}{|G|}\sum_{x\in G}\langle x v_i, v_j\rangle\overline{\langle x w_k, w_e\rangle} = \frac{1}{|G|}\sum_{x\in G}\langle x v_i, v_j\rangle\langle x w_k^*, w_e^*\rangle$$

$$= \langle P(v_i\otimes w_k^*), (v_j\otimes w_e^*)\rangle \quad , \text{ where } P = \frac{1}{|G|}\sum_{x\in G} x$$

P is the projector onto the trivial subrepresentation.

$\Rightarrow P(V\otimes W^*) = 0$ if $V \not\simeq W$, since V, W are irreducible.

$\Rightarrow$ (1): $(t_{ij}^V, t_{kl}^W) = 0$ if $V \not\simeq W$.

$P(V\otimes V^*) = (V\otimes V^*)^G \qquad \dim (V\otimes V^*)^G = \dim \mathrm{Hom}_G(V,V) = 1.$

$V\otimes V^* = V_0 \oplus L$, where $V_0 = \mathrm{span}\left(\sum_k v_k \otimes v_k^*\right)$, $L = \mathrm{span}\left(\sum_{k,l} a_{kl} v_k \otimes v_l^*\right)$ with $\sum a_{kk} = 0$

$\mathrm{Hom}_G(V,V) \simeq \mathrm{span}(Id)$ $\in V\otimes V^*$

$Id: V\to V$ is given by $\sum v_k \otimes v_k^*$: $\sum_k v_k \otimes \underbrace{v_k^*\left(\sum b_i v_i\right)}_{\delta_{ik} b_k} = \sum b_k v_k$

$\Rightarrow P: V\otimes V^* \to V_0 = \mathrm{span}\left(\sum_k v_k \otimes v_k^*\right)$, $\quad P(v_k \otimes v_i^*) = 0$, $i\neq k$

$P^2 = P$: $P(v_i\otimes v_i^*) = \alpha \sum v_k \otimes v_k^*$

$$\left.\begin{aligned} P^2(v_i\otimes v_i^*) &= P\left(\alpha \sum v_k\otimes v_k^*\right) = \alpha\cdot \dim V\cdot \alpha \sum v_k\otimes v_k^* \\ &\parallel \\ P(v_i\otimes v_i^*) &= \alpha \sum v_k \otimes v_k^* \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \alpha^2 \cdot \dim V = \alpha \Rightarrow \alpha = \frac{1}{\dim V}.$$

$$\Rightarrow (2): \quad (t^V_{ij}, t^V_{kl}) = \frac{1}{\dim V}\delta_{ik}\delta_{jl}. \qquad \blacksquare$$

<u>Corollary.</u> $\{t^V_{ij}\}_{V \in \operatorname{Irr} G}$ form an orthonormal basis in $F(G, \mathbb{C})$.

Proof: $\{t^V_{ij}\}_{V \in \operatorname{Irr} G}$ are linearly independent by orthonormality.

$$\dim F(G,\mathbb{C}) = \dim \mathbb{C}[G] = \dim \bigoplus_{V \in \operatorname{Irr} G} \operatorname{End} V = \sum_{V \in \operatorname{Irr} G} (\dim V)^2 = \left|\{t^V_{ij}\}_{V \in \operatorname{Irr} G}\right| \qquad \blacksquare$$

<u>Example.</u> $G = D_4$ $\quad \rho_4(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \rho_4(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \rho_4(sr) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\rho_4(r^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \rho_4(r^3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \rho_4(sr^2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\quad \rho_4(sr^3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$(t^4_{12}, t^4_{12}) = \frac{1}{8}\left(0^2 + 1^2 + 1^2 + 0^2 + 0^2 + (-1)^2 + (-1)^2 + 0^2\right) = \frac{1}{2} = \frac{1}{\dim V}$$

$\rho_3(s) = 1, \; \rho_3(r) = -1 \Rightarrow (t^3, t^4_{12}) = \frac{1}{8}\big(0 \cdot 1 + 1 \cdot (-1) + 0 \cdot 1 + (-1)\cdot(-1) + 1 \cdot 1 + 0 \cdot (-1) \,+$

$+ (-1)\cdot 1 + 0 \cdot (-1)\big) = 0.$