Lecture 5. Recall:

Corollary (Density theorem)

Let V be an irreducible finite dimensional representation of A over an algebraically closed field.
Then the map $\rho: A \to \mathrm{End}\, V$ is surjective.

Matrix algebra: (1) $V \simeq k^n$ is irreducible for $\mathrm{Mat}_n(k)$ (it is cyclic)

(2) $\mathrm{Mat}_n(k) \simeq V^{\oplus n}$ the left regular representation.

(3) Left ideals in $\mathrm{Mat}_n(k) \longleftrightarrow$ subrepresentations in $V^{\oplus n} \Rightarrow$ isomorphic to $V^{\oplus r}$, $r \leq n$. Maximal subrepresentation $\simeq V^{\oplus(n-1)}$

$\Rightarrow$ Any irreducible $\simeq \mathrm{Mat}_n(k) / V^{\oplus(n-1)} \simeq V^{\oplus n} / V^{\oplus(n-1)} \simeq V \simeq k^n$

(cf [PS 4]).

Today: Structure of finite dimensional algebras

Extend the above to the case when V is an irreducible representation of an algebra A. Then $\mathrm{End}\,V$ also has a structure of an A-representation by left multiplication: $\varphi \in \mathrm{End}\,V \Rightarrow$

$$\varphi(v) \in V, \; v \in V \Rightarrow (a\cdot\varphi)(v) = a\,\varphi(v). \qquad \dim V = n.$$

Similarly to above, let $\{v_1 \dots v_n\}$ be the standard basis in V

Then $F: \mathrm{End}\,V \to V^{\oplus n}$, $F(\varphi) = (\varphi(v_1), \varphi(v_2) \dots \varphi(v_n))$

defines an isomorphism of A-representations $\mathrm{End}\,V \simeq V^{\oplus n}$.

Direct sum of algebras.

Def. $A = A_1 \oplus A_2 \oplus \dots \oplus A_n$ direct sum of vector spaces

in particular $1_j \in A_j$ $\forall j$. Then $1 \in A \Rightarrow 1 = 1_1 + 1_2 + \dots + 1_n$

$$a_i a_j = 0 \quad \forall i \neq j, \; a_i \in A_i, \; a_j \in A_j \; ; \text{ In particular } 1_i 1_j = \delta_{ij}.$$

A_i's are two-sided ideals in A.

Proposition. Let $A = A_1 \oplus \dots \oplus A_n$ direct sum of algebras.
Any representation V of A decomposes as a direct sum $V = \bigoplus_{i=1}^{n} V_i$ of subrepresentations, where V_i is a representation of A_i and A_j acts by zero in V_i, $i \neq j$.
If V is irreducible over $A \Rightarrow 1_i V$ is irreducible over A_i for exactly one $i \in \{1, \dots n\}$, and $1_j V = 0$ $\forall j \neq i$.

Proof: Let V be a representation of A. Then $1_i V = V_i$ is a representation of A_i : $\rho(a_i) V_i = \rho(a_i)\rho(1_i) V = \rho(a_i 1_i) V = \rho(a_i) V$

Conversely, if V_i is a representation of $A_i \Rightarrow V_1 \oplus V_2 \oplus \dots \oplus V_n$ is a representation of A by

$$\rho(a_1, a_2 \dots a_n)(v_1 \dots v_n) = (\rho_1(a_1) v_1, \rho_2(a_2) v_2 \dots \rho_n(a_n) v_n)$$

block-diagonal action.

Suppose $v \in V \Rightarrow 1v = v = \sum_{i=1}^{n} 1_i v = \sum_{i=1}^{n} v_i$, where $v_i \in V_i$

$1_j v = \sum_{i=1}^{n} 1_j 1_i v = v_j$ $\Rightarrow$ decomposition of $v = \sum_{i=1}^{n} v_i$ is uniquely determined.

$\Rightarrow$ $V = V_1 \oplus V_2 \cdots \oplus V_n$ the sum is direct. $\Rightarrow$ $V = \bigoplus_{i=1}^{n} 1_i V$

If V is irreducible $\Rightarrow$ $\exists!\, i : 1_i V \neq 0$ $\Rightarrow$ $1_i V = V_i = V$

V_i irreducible over A $\Leftrightarrow$ V_i is irreducible over A_i

Conversely, if $V = V_i \oplus V_j$ more than one summand $\Rightarrow$ we have a nontrivial subrepresentation of A. ∎

<u>Corollary.</u> Irreducible representations of $\bigoplus_{i=1} \mathrm{Mat}_{n_i}(k)$ are $V_1 \simeq k^{n_1}$, $V_2 \simeq k^{n_2}$, … $V_r = k^{n_r}$.

Structure of finite dimensional algebras.

Def. The radical of a finite dimensional algebra A is the set of all elements of A that act by 0 in all irreducible representations of A. Rad A is a two-sided nilpotent ideal [PS5]

Theorem. A finite dimensional algebra A over an algebraically closed field k has only finitely many inequivalent irreducible representations. Each irreducible representation is finite dimensional, and

$$A/\mathrm{Rad}(A) \simeq \bigoplus_{i=1}^{n} \mathrm{End}\, V_i$$

where $\{V_i\}_{i=1}^{n}$ is a complete list of inequivalent irreducible representations of A.

Proof: (1) If V a representation of A, $v \in V$ nonzero vector $\Rightarrow$

$A \curvearrowright V$ is a subrepresentation, $\dim A < \infty \Rightarrow$ every irreducible representation of A is finite dimensional.

(2) $\{V_1, V_2, \dots\}$ inequivalent irreducible representations.

Then $\bigoplus_i \rho_i : A \to \bigoplus_i \operatorname{End} V_i$ is surjective (by an extension of density theorem).

$\Rightarrow$ $n \leq \sum_{i=1}^{n} \dim \operatorname{End} V_i \leq \dim A$ $\Rightarrow$ there can be only finitely many inequivalent irreducible representations.

(3) $\bigoplus_{i=1}^{n} \rho_i : A \longrightarrow \bigoplus_{i=1}^{n} \operatorname{End} V_i$ is surjective.

$$A / \ker(\oplus \rho_i) \simeq \operatorname{Im}(\oplus \rho_i) \simeq \bigoplus_{i=1}^{n} \operatorname{End} V_i$$

$$\ker\left(\bigoplus_{i=1}^{n} \rho_i\right) = \{x \in A : \rho_i(x) = 0 \ \forall i = 1 \dots n\} \stackrel{def}{=} \operatorname{Rad} A$$

$$\Rightarrow \quad A / \operatorname{Rad} A \simeq \bigoplus_{i=1}^{n} \operatorname{End} V_i \qquad \blacksquare$$

<u>Corollary</u> $\dim A = \sum_{i=1}^{n} (\dim V_i)^2 + \dim \operatorname{Rad} A$ for a finite dimensional algebra A over k.

<u>Example.</u> $A = \mathbb{C}[x]/\langle x^n \rangle$ $n \in \mathbb{N}$ fixed. $\{1, x .. x^{n-1}\}$ basis in A

Irreducible representations: $\dim V = 1 \Rightarrow \rho(x) = \lambda \in \mathbb{C}$, $\rho(x^n) = \lambda^n = 0 \Rightarrow \lambda = 0$

$\Rightarrow$ unique irreducible representation V_0 where $\rho(x) = 0$

$$\underset{n}{\underset{\parallel}{\dim A}} = \underset{1}{\underset{\parallel}{1^2}} + \underset{(n-1)}{\underset{\parallel}{\dim \operatorname{Rad} A}} \qquad \operatorname{Rad} A = \langle x \rangle \Rightarrow \dim \operatorname{Rad} A = n-1.$$

Recall: An algebra A is semisimple if every fin dimensional representation of A is completely reducible (decomposes as a direct sum of irreducible representations).

Theorem. (Structure theorem of semisimple finite dimensional algebra).

Let A be a finite dimensional algebra over an algebraically closed field k. Then

$$A \text{ is semisimple} \iff A \simeq \bigoplus_{i=1}^{r} \mathrm{Mat}_{n_i}(k)$$

Proof. ($\Rightarrow$) Consider the left regular representation of A.

A is semisimple $\Rightarrow A = \bigoplus_{i=1}^{r} V_i^{\oplus n_i}$ for some irreducible V_i's.

Consider the intertwiners $\varphi : A \to A$ endomorphisms of A-representations

$$\mathrm{End}_A A = \mathrm{End}_A\left(\bigoplus_{i=1}^{r} V_i^{\oplus n_i}\right)$$

Schur's lemma: $\mathrm{Hom}_A(V_i, V_j) = 0, \; i \neq j$
$= k, \; i = j$

$$\Rightarrow \mathrm{Hom}_A(V_i^{\oplus n_i}, V_i^{\oplus n_i}) \simeq \mathrm{Mat}_{n_i}(k)$$

Ex: $\mathrm{End}_A(V \oplus V) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$

$$\Rightarrow \operatorname{End}_A A \simeq \bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k)$$

$\operatorname{End}_A A \simeq A^{op}$ = A as a vector space with opposite multiplication.

$\varphi \mapsto \varphi(1)$ $\qquad a \overset{op}{\cdot} b = ba \in A. \quad \forall a, b \in A.$ [PS5]

$$\Rightarrow A^{op} \simeq \bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k), \quad (\operatorname{Mat}_n(k))^{op} \simeq \operatorname{Mat}_n(k)$$

$$M \mapsto M^T, \text{ then } (AB)^T = B^T A^T$$

$$\Rightarrow A \simeq \bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k)$$

($\Leftarrow$) Let $A = \bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k)$. First, $\operatorname{Mat}_n(k)$ is a semisimple algebra:

Any fin dim representation of $\operatorname{Mat}_n(k)$ decomposes into a direct sum of irreducible representations ($\simeq V^{\oplus r}$, $V \simeq k^n$). [PS5]

Hint: Take $\{E_{ij}\}_{1 \le i,j \le n}$ a basis in $\operatorname{Mat}_n(k)$, $1 = \sum_{i=1}^{n} E_{ii}$.

$\Rightarrow A = \bigoplus_{i=1}^{r} \operatorname{Mat}_{n_i}(k)$ is semisimple as a direct sum of semisimple algebras. $\blacksquare$

In particular, if A is semisimple fin dimensional, then

$$\operatorname{Rad} A = 0$$

$$A/\operatorname{Rad} A \simeq \bigoplus_{i=1}^{r} \operatorname{End} V_i \quad , \quad A \text{ semisimple} \Rightarrow A \simeq \bigoplus_{i=1}^{r} \operatorname{End} V_i$$

Def. A is simple $\Leftrightarrow$ it is semisimple and has exactly one irreducible representation $\Leftrightarrow$ $A \simeq \operatorname{Mat}_n(k)$ $\Leftrightarrow$ A has only 0 and A as two-sided ideals.

Characters of representations.

Def. Let V be a representation of an algebra A. Then the *character* $\chi_V : A \to k$ is given by

$$\chi_V(a) = \mathrm{Tr}_V\, \rho(a)$$

Remark Clearly $[A,A] = \{xy - yx \,,\, x, y \in A\} \subset \ker \chi_V$:

$$\mathrm{Tr}_V(\rho(x)\rho(y)) = \mathrm{Tr}_V(\rho(y)\rho(x)) \Rightarrow \chi_V : \underbrace{A/[A,A]}_{\text{as a vector space}} \longrightarrow k$$

Theorem. (1) Characters of distinct irreducible finite dimensional representations of A are linearly independent.

(2) If A is a finite dimensional semisimple algebra, then these characters $\{\chi_{V_i}\}_{i=1}^{r}$ form a basis in $(A/[A,A])^*$

Proof: (1) If $V_1 \dots V_r$ are inequivalent irreducible representations of A,
then $\rho_{V_1} \oplus \rho_{V_2} \oplus \dots \oplus \rho_{V_r} : A \to \operatorname{End} V_1 \oplus \dots \oplus \operatorname{End} V_r$ is surjective
(density theorem).

If $\sum_{i=1}^{r} \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A \Rightarrow \sum_{i=1}^{r} \lambda_i \operatorname{tr} M_i = 0$ for all $M_i \in \operatorname{End} V_i$.

In particular $\exists\, a \in A : \rho_{V_i}(a) = \mathrm{Id}_{V_i}$, $\rho_{V_j}(a) = 0$, $i \neq j \Rightarrow$
get $\lambda_i \cdot n = 0 \Rightarrow$ all $\lambda_i = 0$.

$\Rightarrow \{\chi_{V_i}\}_{i=1}^{r}$ are linearly independent. ◿

(to be continued...).