Lecture 4. Recall:

Maschke's theorem Let G be a finite group and k a field s.t. char k does not divide $|G|$.
Then the algebra $k[G]$ is semisimple: If V is a finite dimensional representation of G and $W \subset V$ a subrepresentation, then there exists a subrepresentation $W' \subset V$ s.t. $V \simeq W \oplus W'$ as representations.

Today: More on the structure of the algebra $k[G]$ (Density theorem)

Theorem (Weyl's unitary trick)

Finite dimensional unitary representation of any group is completely reducible.

Today: Weyl's unitary trick for a continuous compact group (Example $U(1)$).

Def. Let V be a complex vector space. A representation (V,ρ) of a group G is unitary if there exist a Hermitian inner product in V such that
$$\langle \rho(g)v, \rho(g)w\rangle = \langle v, w\rangle \quad \forall v,w\in V,\ \forall g\in G.$$

Let G be a compact smooth group (a compact Lie group)

Let (V,ρ) be a complex fin dim representation of G

Starting with any Hermitian inner product $\langle\,,\rangle_0$ in V, we can average it over the action of G to get
$$\langle v, w\rangle = \int_G \langle \rho(g)v, \rho(g)w\rangle_0\, dg\,,$$
where dg is a G-invariant measure on G.

$(dg' = d(gx) = dg \quad \forall x\in G)$

Theorem A compact Lie group has an invariant measure (the Haar measure)

(hard in general).

Then $\langle \rho(x)v, \rho(x)w\rangle = \int_G \langle \rho(gx)v, \rho(gx)w\rangle dg = \int_G \langle \rho(g')v, \rho(g')w\rangle dg' =$

$= \langle v, w\rangle.$

Theorem. Any finite dimensional representation over $\mathbb{C}$ of a compact Lie group is completely reducible.

Proof: same as last time. (it can be turned into a unitary representation)

Example: $U(1)$. $U(1) = S^1$ is the group of rotations in a plane (a circle group)

$U(1) = \{ e^{i\theta}, \ \theta \in [0, 2\pi[\} \subset \mathbb{C}$

Theorem If $\rho: U(1) \to GL(n, \mathbb{C})$ is a representation of $U(1)$ in V, then $V \simeq V_1 \oplus V_2 \oplus \ldots \oplus V_n$, $\rho_i(e^{i\theta}) = e^{im_i\theta}$, $m_i \in \mathbb{Z}$ in V_i

Claim 1. Irreducible unitary representations of $U(1)$ are one-dimensional given by $\{\rho(e^{i\theta}) = e^{in\theta}\}_{n\in\mathbb{Z}}$

Proof: By Schur's lemma, $U(1)$ is abelian $\Rightarrow$ irreducibles over $\mathbb{C}$ are 1-dimensional. $\Rightarrow \rho(e^{i\theta}) = \lambda \in \mathbb{C}^*$

Unitary $\Rightarrow$ $\langle x, y\rangle = \langle \rho(e^{i\theta})x, \rho(e^{i\theta})y\rangle = \langle \lambda x, \lambda y\rangle = \lambda\bar{\lambda}\langle x, y\rangle$

$\Rightarrow |\lambda|^2 = 1$.

Let $\rho(e^{i\theta}) = e^{it(\theta)}$ $\Rightarrow$ $t(0)=0$, $t(\theta_1+\theta_2) = t(\theta_1) + t(\theta_2)$

$$\rho(e^{i\theta_1}e^{i\theta_2}) = e^{it(\theta_1)}e^{it(\theta_2)} = e^{it(\theta_1+\theta_2)}$$

$\Rightarrow$ $t(\theta) = \alpha\theta$ linear function;

$$\rho(e^{i(\theta+2\pi)}) = \rho(e^{i\theta}) \Rightarrow e^{i\alpha\theta} = e^{i\alpha(\theta+2\pi)} = e^{i\alpha\theta}e^{i2\pi\alpha}$$

$\Rightarrow \alpha \in \mathbb{Z}$.

$\Rightarrow$ $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ the list of inequivalent irreducible representations of $U(1)$ $\blacksquare$

Claim 2. V admits a $\mathcal{U}(1)$-invariant hermitian inner product.

Proof: Let $\langle\,,\,\rangle_0$ be any Hermitian inner product on V =>

Set $\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \rho(e^{i\theta})u, \rho(e^{i\theta})v\rangle_0\, d\theta$ =>

$\mathcal{U}(1)$-invariant measure

$$\langle \rho(e^{i\xi})u, \rho(e^{i\xi})v\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \rho(e^{i(\theta+\xi)})u, \rho(e^{i(\theta+\xi)})v\rangle_0\, d(\theta+\xi) = \langle u, v\rangle.$$

ξ is fixed $\Rightarrow d(\theta+\xi) = d\theta$ $\blacksquare$

Therefore, any complex fin dim representation V of $\mathcal{U}(1)$ is unitary => completely reducible by the Weyl trick

=> $V \simeq \bigoplus_{i=1}^{n} V_i$, where $\rho_i(e^{i\theta}) = e^{im_i\theta}$, $m_i \in \mathbb{Z}$

Algebras, modules, ideals.

Def. Left A-module structure on M is given by a map $A \times M \to M$
k-algebra (A), k-vector space (M)

$(a, m) \to am \in M$ that satisfies:

$\forall a, b \in A$

$\forall m, n \in M$

(1) $(ab)m = a(bm)$ $\longleftrightarrow$ $\rho(ab) = \rho(a)\rho(b)$

(2) $1m = m$ $\longleftrightarrow$ $\rho(1) = Id$

(3) $(a+b)m = am + bm$ $\longleftrightarrow$ $\rho(a+b) = \rho(a) + \rho(b)$

(4) $a(m+n) = am + an$ $\longleftrightarrow$ $\operatorname{End} M$ linear transformations.

$\left.\begin{array}{l}(1)\\(2)\\(3)\end{array}\right\} \; \rho : A \to \operatorname{End} M$ alg homomorphism

This is equivalent to defining a representation $(\rho(A), M)$

Def. $I \subset A$ is a left ideal if I is a vector subspace and $aI \subset I$ $\forall a \in A$

$I \subset A$ right $Ia \subset I$ $\forall a \in A$

$I \subset A$ two-sided ideal if it is a left and right ideal.

$J \subset A$ is a maximal ideal if there is no ideal $I \subset A$:

$$J \subsetneq I \subsetneq A.$$

$A/I = \{a+I\}$ is a set of cosets of I

Def. A representation V of A is cyclic if $\exists$ a vector $v \in V$ that generates V: $\rho(A)v = V$.

Proposition. (1) $I \subset A$ is a left ideal $\Leftrightarrow$ $I \subset A$ is a subrepresentation of the left regular representation of A and A/I a quotient representation

$\forall a \in I: \quad a \cdot I \subset I$

(2) V is cyclic $\Leftrightarrow$ $V \simeq A/I$ for some left ideal $I \subset A$ [PS4]

(3) V is irreducible $\Leftrightarrow$ every nonzero $v \in V$ is cyclic

V irreducible $\Rightarrow$ $v \in V$ $\Rightarrow$ $Av \subset V$ subrepresentation $\Rightarrow$ $Av = V$

$W \subset V$ not irreducible $\Rightarrow$ $w \in W \Rightarrow Aw \subset W \neq V \Rightarrow w$ not cyclic

(4) V is irreducible $\Leftrightarrow$ $V \simeq A/I_{max}$ for a maximal left ideal

If $I \subsetneq J \subsetneq A \Rightarrow J/I \subset A/I$ is a subrepresentation

$$\left\{ A/I \right\} \xleftrightarrow{\sim} \left\{ \text{Cyclic } A\text{-representations} \right\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\left\{ A/I_{max} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} V \text{ cyclic: } \forall v \in V, \\ v \neq 0 \text{ is cyclic} \end{array} \right\} \xleftrightarrow{\sim} \left\{ V \text{ irreducible} \right\}$$

Example. Let $A = Mat_n(\mathbb{C})$. Then any irreducible representation of A is isomorphic to $\mathbb{C}^n$ [PS4]

$$\begin{pmatrix} & & \\ & * & \\ & & \end{pmatrix} \begin{pmatrix} & * & \\ & * & \\ 0 & * & 0 \\ & * & \\ & * & \end{pmatrix} = \begin{pmatrix} & * & \\ & * & \\ 0 & * & 0 \\ & * & \\ & * & \end{pmatrix} \Rightarrow$$

$\Rightarrow Mat_n(k) \simeq \bigoplus_{i=1}^{n} \mathbb{C}^n$ as a left regular representation

Density theorem.

Lemma Let V_i $1 \le i \le m$ be irreducible fin dimensional nonisomorphic representations of an algebra A over an algebraically closed field.

Let $V = \bigoplus_{i=1}^{n} V_i^{\oplus n_i}$. Suppose $W \subset V$ is a subrepresentation.

Then $W \simeq \bigoplus_{i=1}^{n} V_i^{\oplus r_i}$, $r_i \le n_i$ $\forall i$ and the inclusion $\varphi_i : V_i^{\oplus r_i} \to V_i^{\oplus n_i}$

is given by a constant $r_i \times n_i$ matrix X_i

$$\varphi_i(v_1 \dots v_{r_i}) = (v_1 \dots v_{r_i}) \underbrace{\begin{pmatrix} & & \\ & X_i & \\ & & \end{pmatrix}}_{n_i} \Big\} r_i = (u_1 \dots u_{n_i}) \in V_i^{\oplus n_i}$$

Proof: By induction on $n = \sum_{i=1}^{m} n_i$.

By Schur's lemma, $V_j \to V_i$, $i \neq j$ is zero, so we only need to consider the inclusion of isotypical components

Base: $n = 1 \Rightarrow V = V_i \Rightarrow \underset{\text{nonzero}}{W} \subset V_i \Rightarrow W \simeq V_i$

Induction step: Let $P \subset W$ irreducible. Then $P \simeq V_i$

$$\varphi(P) \hookrightarrow V_i^{\oplus n_i} : \quad v \to (v q_1, \dots v q_{n_i}) \qquad \text{where } q_i \in k$$

Ex:
$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \to \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \oplus \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

$$V_i \qquad V_i \oplus V_i$$

$$\varphi(v_0) = \alpha u_0 + \beta w_0$$

$$\varphi(g v_0) = \alpha g u_0 + \beta g w_0$$

$$\varphi(v_1) = \alpha u_1 + \beta w_1$$

Here $X = (\alpha \;\; \beta)$

Choose $F \in GL_{n_i}(k)$ such that $(q_1 \dots q_{n_i}) F = (1\,0\,0 \;\; 0)$

F acts on $V_i^{\oplus n_i}$ by $(v_1 \dots v_{n_i}) \to (v_1 \dots v_{n_i}) F$

$$P \xhookrightarrow{\varphi_i} V_i^{\oplus n_i} \xrightarrow{F} V_i^{\oplus n_i} F \simeq V_i \oplus W'$$

Use the induction hypothesis for W' with $n-1$ total irreducible components.

$\Rightarrow$ By induction, get an inclusion given by an $r_i \times n_i$-matrix X_i $\blacksquare$

Theorem (density theorem). Let V be an irreducible finite dimensional representation of A over an algebraically closed field. Let $\{v_1 \dots v_n\} \in V$ linearly independent vectors. Then for any set $\{w_1 \dots w_n\}$ of vectors in V, there exist $a \in A$ such that $\rho(a) v_i = w_i \quad \forall i$.

Proof. Assume the contrary. Then $T: A \to V^{\oplus n}$

$$a \to (\rho(a)v_1, \rho(a)v_2 \dots \rho(a)v_n)$$

is such that $\operatorname{Im} T \subset V^{\oplus n}$ is a proper subrepresentation.

$\Rightarrow$ the inclusion $\underbrace{\operatorname{Im} T}_{\simeq V^{\oplus r}} \hookrightarrow V^{\oplus n}$ is given by a matrix $X = \left(\quad \right)_n^r$ where $r < n$.

by the Lemma

Let $a = 1 \Rightarrow \operatorname{Im} T \ni (v_1, v_2 \dots, v_n) \Rightarrow$

$$\exists \ \underbrace{(u_1 \dots u_r)}_{\in V^{\oplus r}} X = \underbrace{(v_1, v_2 \dots v_n)}_{\in \operatorname{Im} T \subset V^{\oplus n}}$$

Let $(t_1 \dots t_n) \neq 0$ a vector s.t. $\begin{pmatrix} & & \\ & X & \\ & & \end{pmatrix} \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

because $r < n$

$$\Rightarrow \sum_{i=1}^{n} t_i v_i = (v_1 \dots v_n) \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = (u_1 \dots u_r) \underbrace{\begin{pmatrix} & & \\ & X & \\ & & \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}}_{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}} = 0.$$

$\Rightarrow$ contradiction since $\{v_i\}_{i=1}^{n}$ are linearly independent. $\blacksquare$

Corollary (Density theorem)

Let V be an irreducible finite dimensional representation of A over an algebraically closed field.

Then the map $\rho : A \to \operatorname{End} V$ is surjective.

Proof: Let $\varphi \in \operatorname{End} V$, and $\{v_i\}_{i=1}^{n}$ a basis in V.

Let $\{\varphi(v_1)\ \varphi(v_2) \dots \varphi(v_n)\} = \{w_i\}_{i=1}^{n}$. Then $\exists\, a \in A$ s.t. $\rho(a) v_i = w_i$ by the above theorem.

$\Rightarrow \rho(a) = \varphi \in \operatorname{End} V.$ $\blacksquare$