Lecture 3.

Recall: Schur's lemma:

(1) Let V_1 and V_2 be irreducible representations of an algebra A, and $\varphi: V_1 \to V_2$ a nonzero intertwiner. Then φ is an isomorphism and $V_1 \simeq V_2$ as representations of A.

(2) Let V be an irreducible finite dimensional representation of A over an algebraically closed field k, and $\varphi: V \to V$ an intertwiner. Then $\varphi = \lambda \mathrm{Id}_V$ for some $\lambda \in k$.

Today: complete reducibility for $\mathbb{C}$-representations of finite groups (and related questions)

A representation V of A is completely reducible if it is isomorphic to a direct sum of irreducible representations.

$$V \simeq W_1 \oplus \dots \oplus W_n \quad , \; W_i \text{ irreducible}$$

Def An algebra A is semisimple over k if every finite dimensional representation of A over k is completely reducible

Maschke's theorem. Let G be a finite group and k a field s.t. char k does not divide $|G|$.

Then the algebra $k[G]$ is semisimple: If V is a finite dimensional representation of G and $W \subset V$ a subrepresentation, then there exists a subrepresentation $W' \subset V$ s.t. $V \simeq W \oplus W'$ as representations.

Proof: Choose $\hat{W}$: $W \oplus \hat{W} = V$ any complement and let $P : V \to W$, $\hat{W} \to 0$ be the projector along $\hat{W}$.

as vector spaces

Let $\bar{P} = \frac{1}{|G|}\sum_{g\in G}\rho(g)\,P\,\rho(g^{-1})$ where $\rho : k[G] \to \operatorname{End} V$, let $W' = \ker \bar{P}$.

Then $x \in W \Rightarrow \bar{P}(x) = \frac{1}{|G|}\sum_{g\in G}\rho(g)\underbrace{\overset{Id_W}{P}\underbrace{\rho(g^{-1})(x)}_{\in W}}_{\in W} = \frac{1}{|G|}\sum_{g\in G} 1\cdot x = \frac{|G|}{|G|}x = x$

$\Rightarrow \bar{P}|_W = Id_W$

if $y \in V \Rightarrow \bar{P}(y) = \frac{1}{|G|}\sum_{g\in G}\underbrace{\rho(g)\underbrace{P\underbrace{\rho(g^{-1})y}_{\in V}}_{\in W}}_{\in W} \in W \Rightarrow \bar{P}^2 = \bar{P}$ is a projector onto W.

$\in W$

If $z \in W' = \ker \bar{P} \Rightarrow \rho(h)z \overset{?}{\in} W'$

$\bar{P}\rho(h)z = \frac{1}{|G|}\sum_{g\in G}\rho(g)\,P\,\rho(\underbrace{g^{-1}h}_{})z = \frac{1}{|G|}\sum_{k\in G}\rho(hk)\,P\,\rho(k^{-1})z = \rho(h)\underbrace{\bar{P}(z)}_{0,\ z\in W'} = 0$

$k^{-1} \in G \Rightarrow k^{-1} = g^{-1}h \quad g = hk$

$g^{-1} = k^{-1}h^{-1}$

$\Rightarrow W' \subset V$ is a subrepresentation of G $\Rightarrow V = W \oplus W'$ direct sum of subrepresentations $\blacksquare$

Def. Let V be a representation of A and $W \subset V$ a subrepresentation

Let V/W be the quotient space of W-cosets: $v + W$, $v \in V$.

Then V/W is a representation of A by setting $\rho(x)(v+W) = \rho(x)v + W$

If $u \in v + W \Rightarrow \rho(x)(u+W) = \rho(x)(u) + W$ well defined because $\rho(x)(u-v) \in W$. $\forall x \in A$.

Remark. Since representations of G over k: char k does not divide $|G|$ are completely reducible, any quotient representation V/W of G over k is also a subrepresentation: $W' \simeq V/W$ and $V \simeq W \oplus W'$!

In particular, $k[G]$ the left regular representation is isomorphic to a direct sum of irreducible representations of G.

<u>Remark.</u> If G is a finite group and char k does not divide $|G|$, then any irreducible representation of G occurs as a direct summand in the left regular representation $k[G]$. (Exercise).

<u>Example.</u> $A = \mathbb{C}[x]$. All irreducibles are one-dimensional by Schur.

$$\rho(x) = \lambda \in \mathbb{C} \Rightarrow V_\lambda \text{ pairwise non-isomorphic } \{V_\lambda\}_{\lambda \in \mathbb{C}}$$

Indecomposables? V of dim n

$$\rho(x) : V \to V \text{ any matrix} \Rightarrow \text{classified by Jordan normal forms}$$

$$\text{Indecomposables} \longleftrightarrow \rho(x) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = J_{n,\lambda} \quad \text{eigenvalue } \lambda \in \mathbb{C}$$

size $n \times n$

Left regular representation $\mathbb{C}[x]$, and $\langle x^3 \rangle = x^3\mathbb{C}[x] \subset \mathbb{C}[x]$

subrepresentation

Let $V \simeq \mathbb{C}[x] / x^3\mathbb{C}[x]$ the quotient representation

$$\dim V = \{1, x, x^2\}$$

$$\rho_{\text{on } V}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = J_{0,3} \text{ indecomposable}$$

The subrepresentation $\{x^2\} = W$

V is indecomposable, $W \subset V$ not irreducible

$\Rightarrow$ $\mathbb{C}[x]$ is not semisimple.

$$\underbrace{\mathbb{C}x^2}_{\text{irreducible}} \subset \mathbb{C}\{x, x^2\} \subset V.$$

indecomposables

Converse to Maschke's theorem

Let k be a field and G a finite group. If any finite dimensional representation of G is completely reducible, then char k does not divide $|G|$.

Proof. $k[G] = V$ left regular

Let $\varphi : V \to k$ linear map, $\varphi(g) = 1$ $\forall g \in G$. homomorphism $k[G] \to k$ trivial

Then $\ker\varphi$ is G-invariant (recall the proof of Schur's lemma).

Suppose $V = k[G]$ is completely reducible $\Rightarrow$ $V = \ker\varphi \oplus \mathcal{U}$

$$\ker\varphi = \{g-1\}_{\substack{g\in G\\ g\neq 1}} \text{ a basis} \quad ; \quad \dim V = |G|, \quad \dim(\ker\varphi) = |G|-1$$

$\Rightarrow \dim\mathcal{U} = 1 \Rightarrow \mathcal{U} = \{u\}$ spanned by a single vector $u \in k[G]$

$$u = \sum \mu_g g \Rightarrow \underbrace{\underbrace{\rho(x)\cdot u}_{\in\mathcal{U}} - u}_{\in\mathcal{U}} = \sum_{g\in G}\mu_g xg - \sum_{g\in G}\mu_g g = \sum_{g\in G}\mu_g \overbrace{\underbrace{(xg-g)}_{\varphi:\ \downarrow\ 0}}^{\in\ker\varphi} \in \ker\varphi \cap \mathcal{U} = 0$$

$\mu_g \in k$

$\Rightarrow \rho(x)u = u$

Coef. of 1 in u: $\overset{\text{in } \mathcal{U}}{=} \mu_1 \overset{\text{in } \rho(x)u}{=} \mu_{x^{-1}}$ $\forall x \in G \Rightarrow u = \mu_1 \sum_{g\in G} g$

Then $\varphi(u) = \mu_1 \varphi\left(\sum_{g\in G} g\right) = \mu_1 |G| \neq 0$ since $u \notin \ker\varphi$.

$\Rightarrow |G| \neq 0$ in k. $\boxed{/\!/}$

<u>Example</u> $G = C_3 = \{t : t^3 = 1\}$, over $\mathbb{F}_3$

$\mathbb{F}_3[C_3]$ regular representation : in the basis $\{1, t, t^2\}$

$$\rho(t) = \begin{pmatrix} 0 & & 1 \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix} \Rightarrow -\lambda^3 + 1 = 0 \Rightarrow \lambda = 1 \text{ in } \mathbb{F}_3$$ the only eigenvalue.

The only eigenvector $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^{=v}$ – subrepresentation.

Find $(A - \lambda I)^2 v' = 0 \Rightarrow v' = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; $(A - \lambda I)^3 v'' = 0 \Rightarrow v'' = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

generalized eigenvectors

$= \{v\} < \{v, v'\} < \{v, v', v''\} = V = \mathbb{F}_3[C_3]$ indecomposable

$$\rho(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ in the basis $\{v, v', v''\}$.

$\Rightarrow \mathbb{F}_3[C_3]$ is not semisimple.

Another viewpoint on complete reducibility: *The Weyl unitary trick.*

Def. A representation ρ of G in a $\mathbb{C}$-vector space V is unitary if V has a hermitian inner product invariant under the action of G:

$$\langle v, w\rangle = \langle \rho(g)v, \rho(g)w\rangle \quad \forall v, w \in V, \ \forall g \in G.$$

Proposition. Let $\rho : G \to GL(V)$ be a complex representation of a finite group. Then there exists an inner product $\langle\ ,\ \rangle$ on V that is G-invariant.

Proof. Let $\langle\ ,\ \rangle_0$ any hermitian inner product in V

Then $\langle u, v\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \rho(g)u, \rho(g)v\rangle_0$ is G-invariant

$$\langle \rho(h)u, \rho(h)v\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \rho(hg)u, \rho(hg)v\rangle_0 = \langle u, v\rangle.$$

$\boxed{///}$

$\Rightarrow$ Any $\mathbb{C}$-representation of a finite group is unitary.

Theorem (Weyl's unitary trick)

Finite dimensional unitary representation of any group is completely reducible.

Proof: Let V be a unitary representation of G and W a subrepresentation $W < V$.

Let $v \in W^\perp$ wrt the G-ivariant inner product, $\forall w \in W$.

Then $\langle \rho(g)v, w\rangle = \langle \rho(g)v, \rho(g)\rho(g^{-1})w\rangle \overset{G\text{-invariance}}{=} \langle v, \underbrace{\rho(g^{-1})w}_{\in W}\rangle = 0$

$\Rightarrow \rho(g)v \in W^\perp$ for any $v \in W^\perp$, $g \in G$. $\Rightarrow W^\perp \subset V$ is a subrepres.

$\Rightarrow V = W \oplus W^\perp$ as G-representations.

Continuing with $W^\perp$ and W terminates in an irreducible decomposition, since V is finite dimensional. $\blacksquare$

Moreover, we get an orthogonal decomposition into the irreducible components.

Conclusion: Together with the result that any $\mathbb{C}$-representation of a finite group is unitary, this provides an alternative proof of complete reducibility of finite dimensional complex representations of a finite group.