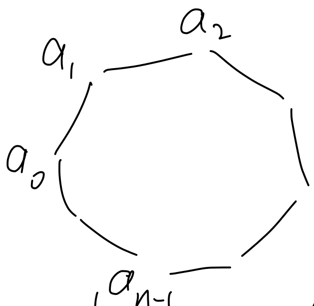
Lecture 2

Recall example 1:



$$M(a_i) = \frac{1}{2}(a_{i+1} + a_{i-1})$$

Let $\{f_i\}_{i=0}^{n-1}$ functions on the vertices, $f_i(a_j) = \delta_{ij}$

Let $R: f_i \to f_{i+1}$ representation of the cyclic group $C_n = \langle t,\ t^n = 1 \rangle$

$L: f_i \to f_{i-1}$ (mod n)

$$R = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}$$ in the basis $\{f_i\}$; $\det(R - \lambda I) = \lambda^n - 1 \Rightarrow$ eigenvalues $= \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$

where $\zeta = e^{\frac{2\pi i}{n}}$

$$L = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 1 \;\; 0 \end{pmatrix}$$; eigenvectors (irreducible representations)

$$\begin{array}{cccccc} & v_0 & v_1 & v_2 & v_{\frac{n}{2}} & v_{n-1} \\ & \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^{n-1} \end{pmatrix} & \begin{pmatrix} 1 \\ \zeta^2 \\ \zeta^4 \\ \vdots \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ \zeta^{n-1} \\ \vdots \\ \vdots \end{pmatrix} \\ & & & & \text{if } n \text{ even} & \\ \text{eigenvalues } R & 1 & \zeta & \zeta^2 & (-1) & \zeta^{n-1} \\ L & 1 & \zeta^{n-1} & \zeta^{n-2} & & \zeta \end{array}$$

We have: $M = \frac{1}{2}(L + R)$ acting on $V = \{f_i\}_{i=0}^{n-1} = \{v_i\}_{i=0}^{n-1}$

$$Mv_j = \frac{1}{2}(L+R)v_j = \frac{1}{2}\left(\xi^{j}+\xi^{-j}\right)v_j = \cos\frac{2\pi j}{n}\, v_j \quad \Rightarrow \qquad \alpha = \frac{2\pi}{n}$$

Let $F = b_0 v_0 + b_1 v_1 + \dots + b_{n-1}v_{n-1} = a_0 f_0 + a_1 f_1 + \dots + a_{n-1} f_{n-1}$ be the starting function

$$M^k(F) = b_0 v_0 + b_1 \cos^k\alpha\, v_1 + b_2 \cos^k 2\alpha \cdot v_2 + \qquad + \underbrace{b_{\frac{n}{2}}(-1)^k v_{\frac{n}{2}}}_{\text{if } n \text{ even}} \qquad + b_{n-1}\cos^k(n-1)\alpha\, v_{n-1}$$

$k \to \infty$:

$b_0 v_0 \qquad 0 \qquad 0 \qquad\qquad\qquad 0$ if n odd

$b_0 v_0 \qquad + \quad (-1)^k b_{\frac{n}{2}} v_{\frac{n}{2}}$ if n even

Change of basis $\Rightarrow$ $b_0 = \frac{1}{n}\sum a_i$; $b_{\frac{n}{2}} = \frac{1}{n}\left(\sum_{odd} a_i - \sum_{even} a_i\right)$

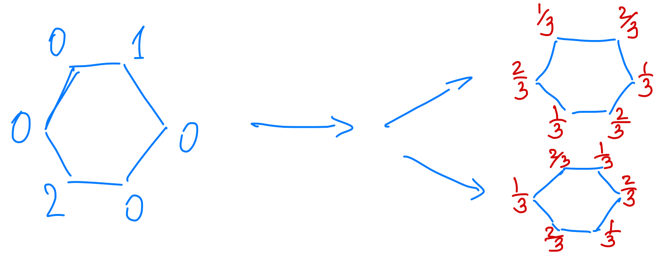
$\Rightarrow$ If n is odd, $M^k F \xrightarrow[k\to\infty]{}$ all vertices $= \frac{1}{n}\sum a_i$

If n is even, $M^k F \xrightarrow[k\to\infty]{}$ at odd vertices: $\frac{2}{n}\sum_{odd} a_i$

at even vertices: $\frac{2}{n}\sum_{even} a_i$

alternating.

Exercise:



alternating

Recall:

Def. $\rho: A \longrightarrow \mathrm{End}_k V$ representation of an associative algebra
algebra homomorphism

Def. $\rho: G \longrightarrow GL(V)$ representation of a group
group homomorphism

$$\mathrm{Rep}\, G \longleftrightarrow \mathrm{Rep}\, k[G]$$

Subrepresentation: $W \subset V$ such that $\rho(a)W \subset W \quad \forall a \in A$

Irreducible representation: $0 \subset V$ and $V \subset V$ are the only subrepres.

Indecomposable representation: $V \neq V_1 \oplus V_2$

Irreducible representation is indecomposable, but the converse is false

Example: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ Jordan normal block: $\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = J_{n,\lambda}$

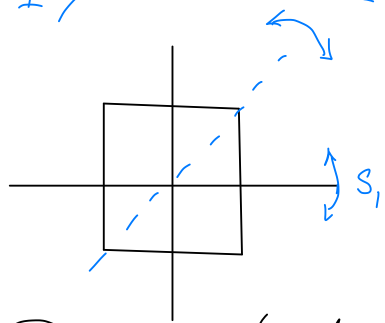
not diagonalizable

The main questions of representation theory:

(1) Classify the irreducible representations of A or G.

(2) Classify the indecomposable representations of A or G.

Example. Consider $D_4 = \langle S_1, S_2 : S_1^2 = 1, S_2^2 = 1, (S_1 S_2)^4 = 1 \rangle$



Irreducible representations of D_4?

Remark. It is sufficient to define ρ(generators) of a group such that they satisfy the relations in G. Then extend by the homomorphism property to all elements of G

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

Start with $\dim V = 1$.

(1) $\rho(S_1) = \lambda_1$ $\rho(S_2) = \lambda_2$ s.t. $\lambda_1^2 = \lambda_2^2 = 1$, $(\lambda_1 \lambda_2)^4 = 1$.

$\Rightarrow \lambda_1, \lambda_2 \in \{\pm 1\} \Rightarrow$ 4 representations

$$\begin{cases} \rho_0(s_1) = \rho_0(s_2) = 1 \\ \rho_1(s_1) = \rho_1(s_2) = -1 \\ \rho_2(s_1) = 1, \ \rho_2(s_2) = -1 \\ \rho_3(s_1) = -1, \ \rho_3(s_2) = 1 \end{cases}$$

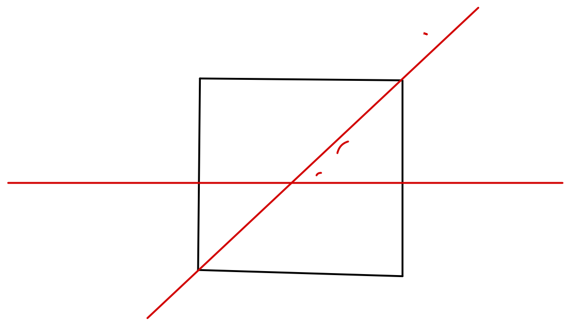
$\Rightarrow \rho(g) = 1 \ \forall g \in D_4$ the trivial representation

$\Rightarrow \rho(\overbrace{s_1 s_2 \ldots}^{\ell}) = (-1)^\ell$ sign representation

They are all pairwise non-isomorphic

$\varphi : \mathbb{C} \to \mathbb{C} \quad : \varphi = \mu \in \mathbb{C}^*$

$\rho_2(\mu s_1) = \mu \rho_3(s_1) \Rightarrow \mu = -\mu \quad \Rightarrow \mu = 0.$ impossible



Suppose $\dim V = 2$.

$\rho : D_4 \to GL(\mathbb{C}^2) \quad : \rho(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rho(s_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ no common invariant subspace $\Rightarrow \rho$ is irreducible in V_2.

Is there another 2-dim irreducible representation?

(1) Eigenvalues for $\rho(s_1), \rho(s_2) \to \pm 1$. If the same, $\Rightarrow \rho(s_1) = \lambda\, Id$ or $\rho(s_2)$

$\Rightarrow$ not irreducible $\Rightarrow$ must have $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for each s_1, s_2 in some basis.

$\Rightarrow \det \rho(s_1) = -1 = \det \rho(s_2)$

$\Rightarrow \det \rho(s_1 s_2) = 1$;

(2) Eigenvalues of $\rho(s_1 s_2)$: $\rho(s_1 s_2)^4 = 1 \Rightarrow \{\pm i, \pm 1\}$, $\det \rho(s_1 s_2) = 1 \Rightarrow$

$\Rightarrow \rho(s_1 s_2) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}$; $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ if $\to$ $(s_1 s_2)$ acts as a scalar $\Rightarrow$ not irreducible

$\Rightarrow$ eigenvalues for $\rho(s_1 s_2)$: i and $-i$ $\Rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ is a rotation by $\frac{\pi}{2}$;

Both s_1 and s_2 are reflections.

$\Rightarrow$ up to a basis change $\rho(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\rho(s_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is equivalent to this representation. Later we will see that (V_0, V_1, V_2, V_3, V) for D_4 is a complete list of irreducible inequivalent repr.

Schur's lemma.

Proposition (Schur's lemma) Let V_1, V_2 be representations of an algebra A over any field.

Let $\varphi: V_1 \to V_2$ be a $\boxed{\text{nonzero}}$ homomorphism of representations.

Then
(1) If V_1 is irreducible, then φ is injective
(2) If V_2 is irreducible, then φ is surjective
(3) If V_1 and V_2 are irreducible, then φ is an isomorphism of irreducible representations.

Proof (1) V_1 irreducible. $\ker\varphi \subset V_1$ is a subrepresentation:

Let $v \in \ker\varphi$, $a \in A$ $\Rightarrow$ $\varphi(\rho_1(a)v) = \rho_2(a)\underbrace{\varphi(v)}_{=0} = 0 \Rightarrow \rho_1(a)v \in \ker\varphi$ if $v \in \ker\varphi$

V_1 irreducible $\Rightarrow$ $\ker\varphi = \begin{cases} 0 \\ V_1 \Rightarrow \varphi(V_1) = 0 \text{ impossible since } \varphi \text{ is nonzero} \end{cases}$

$\Rightarrow \ker\varphi = 0$, $\Rightarrow \varphi: V_1 \to V_2$ is injective

(2) V_2 irreducible : $\operatorname{Im}\varphi \subset V_2$ is a subrepresentation:

if $u \in \operatorname{Im}\varphi \Rightarrow \exists v \in V_1 : \varphi(v) = u \Rightarrow \rho_2(a)\cdot u = \rho_2(a)\cdot\varphi(v) =$

$$= \varphi(\rho_1(a)v) \in \operatorname{Im}\varphi \subset V_2$$

$\Rightarrow \operatorname{Im}\varphi = \begin{cases} 0 & \Rightarrow \varphi(V_1) = 0 \text{ impossible since } \varphi \text{ is nonzero} \\ V_2 & \end{cases}$

$\Rightarrow \operatorname{Im}\varphi = V_2 \Rightarrow \varphi$ is surjective.

(3) V_1, V_2 both irreducible $\Rightarrow$ (1) and (2) show that φ is injective and surjective $\Rightarrow$ isomorphism $\varphi : V_1 \simeq V_2$. ▨

Corollary 1 If V_1, V_2 are irreducible representations of an algebra A and $\dim V_1 \neq \dim V_2$, there is no nonzero homomorphism between them.

Corollary 2. (Schur's lemma over an algebraically closed field).

Let V be an irreducible finite dimensional representation of A over an algebraically closed field k, and $\varphi: V \to V$ be an intertwiner.

Then $\varphi = \lambda \, Id_V$ for some $\lambda \in k$.

Proof: Since k is alg closed $\Rightarrow$ $\varphi: V \to V$ has an eigenvalue $\lambda \in k$.

Then $(\varphi - \lambda Id): V \to V$ commutes with the action of A

$\Rightarrow (\varphi - \lambda Id)$ is an intertwiner, V is irreducible $\Rightarrow$ By Schur's lemma

$$\text{or} \left[\begin{array}{l} \varphi - \lambda Id = 0 \\ \varphi - \lambda Id : V \to V \text{ is an isomorphism. impossible since } \det(\varphi - \lambda Id) = 0 \end{array} \right.$$

$\Rightarrow \varphi = \lambda Id.$ $\boxed{/\!/\!/}$

Remark. Schur's lemma over algebraically closed fields stays true for countably-dimensional representations: if V is an irreducible countably dimensional representation, $\varphi: V \to V$ an intertwiner $\Rightarrow$ φ is a scalar operator.

Remark. Corollary 2 fails over non-alg closed fields in general.

Ex. $A = \mathbb{C}$ as an $\mathbb{R}$-algebra, $V = \mathbb{C}$ a representation of A.

Then V is irreducible, If $\{az\}_{a\in\mathbb{R}}$ is not invariant wrt $\mathbb{C}$-action

But $\varphi: \mathbb{C} \to \mathbb{C}$ an intertwiner does not have to be a real multiplication: any $x \in \mathbb{C}^*$: $x: \mathbb{C} \to \mathbb{C}$ is an intertwiner.

Corollary 3. Let A be a commutative algebra (G an abelian group) over an algebraically closed field k.
Then every fin. dim. irreducible representation over k of A (or G) is one-dimensional

Proof. Let $v \in V \Rightarrow \rho(a)(\rho(b)v) = \rho(ab)v \underset{ab=ba}{=} \rho(ba)v = \rho(b)(\rho(a)v)$ $\forall a,b\in A$

$\Rightarrow \varphi = \rho(a): V \to V$ is an intertwiner

$\Rightarrow$ By Schur's lemma $\rho(a) = \lambda\, \mathrm{Id}_V$ $\forall a \in A \Rightarrow$ every subspace in V is A-invariant $\Rightarrow$ If V is irreducible $\Rightarrow V$ is 1-dimensional. ▨

Ex. Irreducible representations of $C_n = \langle t : t^n = 1 \rangle$ cyclic group.

Abelian group $\Rightarrow$ all irreducibles are 1-dimensional

$\Rightarrow$ find $\lambda \in \mathbb{C}^*$: $\quad \rho(t) = \lambda \Rightarrow (\rho(t))^n = \lambda^n = 1$.

$\Rightarrow \lambda \in \{1, \xi, \xi^2 \dots \xi^{n-1}\}$ where $\xi = e^{\frac{2\pi i}{n}}$. n-th roots of 1.

$\xi^i \neq \xi^j \Rightarrow$ inequivalent representations: $\mu : V_{\xi^i} \to V_{\xi^j}$

$\Rightarrow \mu(\rho_i(t) 1) = \rho_j(t)(\mu \cdot 1) \Rightarrow \mu \xi^i = \xi^j \mu \Rightarrow \xi^i = \xi^j \Rightarrow i = j$

$\Rightarrow n$ inequivalent 1-dimensional irreducible representations. $\{V_{\xi^i}\}_{i=0}^{n-1}$

Consider the regular representation $\rho(t) \cdot \mathbb{C}[C_n] = t \cdot \mathbb{C}[C_n]$

$$t \cdot t^k = t^{k+1}$$

$\Rightarrow$ in the basis $\{1, t \dots t^{n-1}\}$ $\quad \rho(t) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix} \xRightarrow[\text{of basis}]{\text{change}} \begin{pmatrix} 1 & & & 0 \\ & \xi & & \\ & & \xi^2 & \\ 0 & & & \ddots \ \xi^{n-1} \end{pmatrix}$

$\Rightarrow \mathbb{C}[C_n] = \bigoplus_{i=0}^{n-1} V_{\xi^i} \quad ; \rho_i(t) = \xi^i$ in V_{ξ^i} $\quad$ direct sum of all irreducibles each with multiplicity 1.