Representation Theory - I

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All information on Moodle

Math - 314

Lectures : Tuesday 10:15 – 12:00

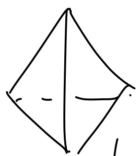
Exercises : Tuesday 8:15 – 10:00

Symetries of objects $\longleftrightarrow$ Groups

(1) Classify all groups

(2) Classify the objects they act on

$\Big\downarrow$ Linearization

$G \circlearrowright$  $\longrightarrow$ instead consider the matrices $M \in End(\mathbb{R}^3)$

by rigid symmetries

This leads to the idea of a linear representation of a group

Instead of studying groups, study homomorphisms $G \to GL(V)$

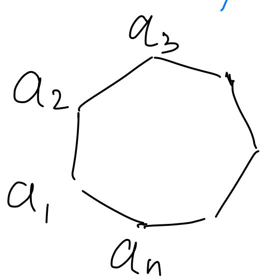
where V is a vector space.

Rep theory: classify the vector spaces where a group can act (linearly).

Motivational examples

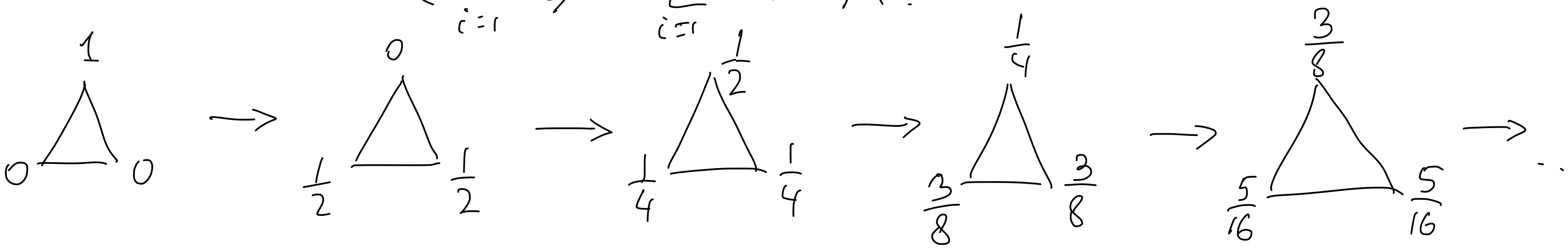
$$M(a_i) = \frac{1}{2}(a_{i+1} + a_{i-1}), \text{ indices mod } n$$

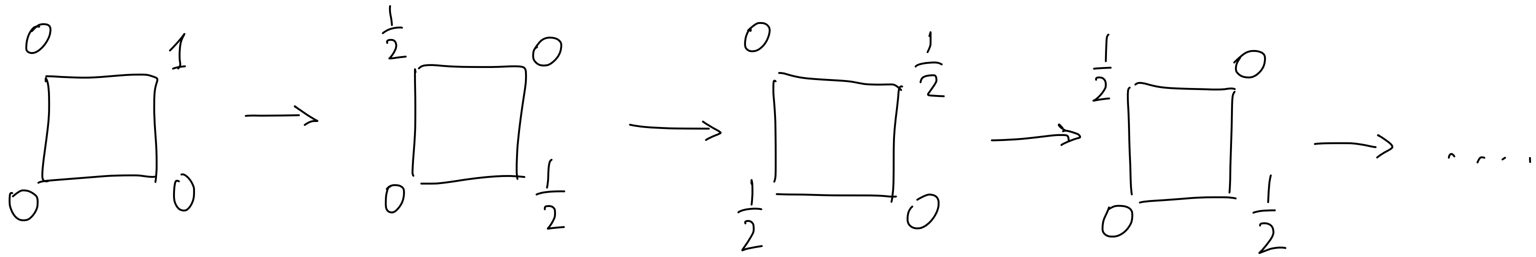
1. „Funny example"



Question: $M^k(a_i) \xrightarrow[k\to\infty]{} ?$

$$M\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} a_i = A.$$





Vector space V, functions on the vertices, basis $\{f_1 \dots f_n\}$: $f_i = \delta_{ij}$

$\circlearrowleft V$ cyclic group acts on it: $R: f_i \to f_{i+1}$

$L: f_i \to f_{i-1}$

operator $M = \frac{1}{2}(L+R)$

Idea: Study the irreducible representations of C_n given by R and L on V, decompose into the irreducible components.

2. „Real life example"

Question: Find the dimension of the space of homogeneous polynomials of degree n in two variables such that

$$\begin{cases} p(u,w) = p(u+w,-u) \\ p(u,w) = -p(w,u) \end{cases} \qquad \{u^n, u^{n-1}w, \dots w^{n-1}u, w^n\} \simeq V \simeq \mathbb{C}^{n+1}$$

$$\underset{b}{\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u+w \\ -u \end{pmatrix} \qquad \underset{a}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} w \\ u \end{pmatrix}.$$

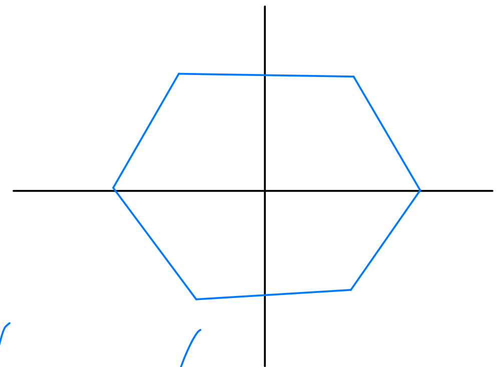
Do they generate a nice group?

$$\Rightarrow a^2 = 1 \;; \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \;; \quad b^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow b^6 = 1$$

$$a b a^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = b^{-1}$$

$$G = \langle a, b : a^2 = 1, b^6 = 1, a b a^{-1} = b^{-1} \rangle = D_6$$

$$\langle S_1, S_2 : S_1^2 = 1, S_2^2 = 1, (S_1 S_2)^6 = 1 \rangle$$



$D_6 \hookrightarrow V$ $\leftarrow$ apply rep. theory of finite gps to this question

In this cours we will consider mostly representations of finite groups in finite dimensional complex vector spaces.
But we will start with a more general setting of algebras.

Def An associative algebra A over a field k is a k-vector space with a bilinear map: $A \times A \to A$ the product
$a, b \to a \cdot b \in A$ such that

(1) $(ab) \cdot c = a(b c)$

(2) $\exists\, 1 \in A : \ 1a = a \cdot 1 = a \quad \forall a \in A$.

Def. A homomorphism of algebras $f : A \to B$ is a k-linear map s.t.

$$\begin{cases} f(ab) = f(a) f(b) & \forall a, b \in A \\ f(1_A) = 1_B \end{cases}$$

f is an isomorphism if $\exists\, g : B \to A$ s.t $f \circ g = Id_B$, $g \circ f = Id_A$
(equivalently, an algebra homomomomorphism that is an isomorphism of vector spaces is an algebra isomorphism).

<u>Examples.</u> (1) $A = k$ 1-dimension

(2) $A = k[x_1, \dots, x_n]$ polynomials in n variables ∞-dimensional

(3) $\mathrm{End}_k V$ linear maps: $V \to V = k^n$; $\mathrm{End}_k V \simeq \mathrm{Mat}_n(k)$

$$\dim(\mathrm{End}_k V) = n^2$$

(4) Free algebra $A = k\langle x_1 \dots x_n \rangle$; elements: k-linear combinations of words in $\{x_1 \dots x_n\}$ letters

Multiplication: concatenation of words ∞-dimensional

(5) Group algebra of a finite group G. $A = k[G]$

elements: $\sum_{g_i \in G} \lambda_i g_i$, $\lambda_i \in k$, $g_i \in G$; basis $= \{g_i\}_{g_i \in G}$

Multiplication: $g, h \to g \cdot h$ group product.

$$\dim k[G] = |G|.$$

Def An algebra is commutative if $ab = ba \quad \forall a, b \in A$.

Examples (1), (2) are commutative, (5) $\iff$ G is commutative;
(3) is commutative $\iff$ $\dim V = 1$; (4) is commutative $\iff$ $n = 1$ $k[x_1] = k\langle x_1 \rangle$.

Def A *representation* of an associative algebra A is a couple (V, ρ) where V is a k-vector space and

$$\rho : A \to \mathrm{End}_k V \text{ is an algebra homomorphism}$$

Explicitly, $\rho : A \to \mathrm{End}_k V$, $\rho(1) = \mathrm{Id}_V$, $\rho(a \cdot b) = \rho(a) \cdot \rho(b) \quad \forall a, b \in A$.

Notations: ρ is often omitted, so we write $\underset{\in A}{a} \cdot \underset{\in V}{v} = w \in V$

Def Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of an algebra A.
A homomorphism $\varphi : V_1 \to V_2$ of representations is a linear map that commutes with the action of A.

$$\varphi(\underbrace{\rho_1(a)\cdot v}_{\in V_1}) = \rho_2(a)\underbrace{\varphi(v)}_{\in V_2} \quad \forall a\in A,\ \forall v\in V_1$$

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(a)"'] & V_2 \arrow[d, "\rho_2(a)"] \\ V_1 \arrow[r, "\varphi"] & V_2 \end{tikzcd}$$

The diagram commutes $\forall a\in A$.

$$\boxed{\varphi\circ\rho_1 = \rho_2\circ\varphi}$$

φ is called an <u>intertwiner</u>

Homomorphisms $(V_1,\rho_1)\to(V_2,\rho_2)$ form a vector space $\mathrm{Hom}_A(V_1,V_2)$.

A homomorphism is an isomorphism of representations if $\exists\ \psi: V_2\to V_1$ homomorphism s.t. $\varphi\circ\psi = id_{V_2}$, $\psi\circ\varphi = id_{V_1}$.

<u>Def.</u> A subrepresentation of (V,ρ) is a subspace $W\subset V$ that is invariant under the action of A: $\rho(a)W\subset W$ $\forall a\in A$.

<u>Examples:</u> $0\subset V$ and $V\subset V$ for any representation V are subrepresentations

Def. A representation of a group G is a couple (V,ρ), where V is a k-vector space and $\rho : G \to GL(V)$ group homomorphism

$$\rho(1) = Id_V$$
$$\rho(g_1 g_2) = \rho(g_1)\cdot\rho(g_2)$$

If $\rho : G \to GL(V)$ is injective, the representation is faithful.

Lemma [PS1] A representation of G is equivalent to a representation of the group algebra $k[G]$.

Examples of group representations

(1) Trivial representation $\quad 1 : G \to GL(V) : g \to Id_V \quad \forall g \in G$

(2) Let G be a finite group, $V = k[G]$ = vector space with basis $\{g\}_{g\in G}$.

Then $\rho : G \to GL(V)$; $\rho(g)h = gh$ is a representation of G called the left regular representation

More generally, if A is an algebra, $\rho_{reg}: A \to \operatorname{End} A$ by $\rho(a) \cdot b = a \cdot b \quad \forall a, b \in A$, the left regular representation of A.

In particular, if $A = k[G] \Rightarrow$ the left regular representation of $k[G]$ on itself is equivalent to the left regular representation of the group G defined above.

<u>Def.</u> A representation (V, ρ) of A (or G) is <u>irreducible</u> if $0 \subset V$ and $V \subset V$ are the only subrepresentations.

<u>Def</u> Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of A.
Then $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ is also a representation of A, defined by

$$\rho(a)(v_1 \oplus v_2) = \rho_1(a) v_1 \oplus \rho_2(a) v_2$$

$$(V, \rho) = \left(\begin{array}{c|c} (V_1, \rho_1) & 0 \\ \hline 0 & (V_2, \rho_2) \end{array}\right)$$

<u>Def</u> A nonzero representation (V, ρ) of A (or G) is <u>indecomposable</u> if it is not isomorphic to a direct sum of two nonzero representations.

In particular, an irreducible representation is indecomposable.

The converse is false in general.