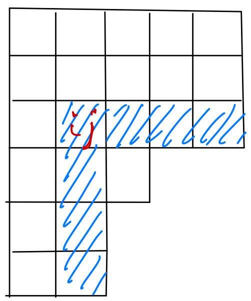
Lecture 14. Recall: representations of S_n.

(1) Complex irreducible representations of S_n are the Specht modules $\{V_\lambda\}_{\lambda \text{ partition of } n}$; $V_\lambda = \mathbb{C}[S_n]c_\lambda$

(2) Character $\chi_{V_\lambda}(C_i)$ = coefficient of $x^{\lambda+\rho} = \prod_j x_j^{\lambda_j+N-j}$ in the polynomial $\Delta x \prod_{m\geq 1} H_m(x)^{i_m}$, $\Delta(x) = \prod_{1\leq i<j\leq N}(x_i - x_j)$

C_i = conj. class of type $(i_1, i_2, \dots i_\ell \dots)$, i_ℓ = # of cycles of length ℓ.

(3) Dimension $\dim V_\lambda = \dfrac{n!}{\prod_{i\in\lambda_j} h(i,j)}$, where $h(i,j)$ is the number of squares in the hook starting at square (i,j) in Y_λ.



(4) Induction and restriction between S_{n-1} and S_n.

(a) $\mathrm{Res}^{S_n}_{S_{n-1}} V_\mu = \bigoplus_{\lambda \in R(\mu)} V_\lambda$, where $R(\mu)$ is the set of Young diagrams obtained by removing one square from Y_μ

(b) $\mathrm{Ind}^{S_n}_{S_{n-1}} V_\mu = \bigoplus_{\lambda \in A(\mu)} V_\lambda$, where $A(\mu)$ is the set of Young diagrams obtained by adding one square to Y_μ.

Example.

$$S_4 \qquad\qquad\qquad S_3$$

$$\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \xrightarrow{\mathrm{Res}} \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\dim = \frac{4!}{4\cdot 2} = 3 \qquad \dim = \frac{3!}{3\cdot 2} = 1 \qquad \dim = \frac{3!}{3\cdot 1} = 2$$

$$3 \quad = \quad 1 \quad + \quad 2$$

$$S_3 \qquad\qquad\qquad S_4$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \xrightarrow{\mathrm{Ind}} \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array} \oplus \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\dim V_\lambda = 2 \qquad \dim = 3 \qquad \dim = \frac{4!}{3\cdot 2\cdot 2} = 2 \qquad \dim = 3$$

$$\dim \mathrm{Ind}^{S_4}_{S_3} V_\lambda = 4\cdot 2 = 8 = 3 + 2 + 3.$$

Schur-Weyl duality

Theorem (Double centralizer)

Let A, B be two subalgebras of $\operatorname{End} E$, where E is a finite dimensional complex vector space. Suppose that A is semisimple and $B = \operatorname{End}_A E$. Then

(1) $A = \operatorname{End}_B E = \operatorname{End}_{\operatorname{End}_A E} E$ (double centralizer)

(2) B is semisimple

(3) As a representation of $A \otimes B$, $E \simeq \bigoplus_{i \in I} V_i \otimes W_i$

where $\{V_i\}_{i \in I}$ are the irreducible representations of A

$\{W_i\}_{i \in I}$ are the irreducible representations of B.

Proof. A semisimple $\Rightarrow E = \bigoplus_{i\in I} V_i \otimes W_i$, where $W_i = \mathrm{Hom}_A(V_i, E)$ the multiplicity space

e.g. $V_i^{\oplus 3} = V_i \otimes \mathbb{C}^3$, $W_i = \mathbb{C}^3$, $W_i = \mathrm{Hom}_A(V_i, E)$, here W_i is (1×3)-matrix. from the Density thm.

$$B = \mathrm{End}_A E = \mathrm{Hom}_A\left(\bigoplus_{i\in I} V_i\otimes W_i, \bigoplus_{i\in I} V_i \otimes W_i\right) \simeq \bigoplus_{i\in I} \mathrm{End}(W_i)$$ – direct sum of matrix alg.

$$\mathrm{Hom}_A(V_i, V_j) = \delta_{ij}\,\mathbb{C}$$

$\Rightarrow B$ is semisimple, and $\{W_i\}_{i\in I}$ is the complete set of the irreducible representations.

$$\underline{\mathrm{End}_B E} = \mathrm{Hom}_B\left(\bigoplus_{i\in I} V_i\otimes W_i, \bigoplus_{i\in I} V_i\otimes W_i\right) \simeq \bigoplus_{i\in I} \mathrm{End}(V_i) = \underline{A}$$

and $\{V_i\}_{i\in I}$ is the complete set of its irreducible representations.

$$\Rightarrow \underline{E = \bigoplus_{i\in I} V_i \otimes W_i}$$

$\blacksquare$

Idea: Apply this to $E = V^{\otimes n}$ where $A = \mathbb{C}[S_n]$ semisimple acting by the permutation of factors, and $B = \mathrm{End}_A V^{\otimes n}$.

Consider $\mathrm{End}(V) = \mathrm{Mat}_k(\mathbb{C})$ $\dim V = k$

Introduce operation $[A,B] = AB - BA$ on $\mathrm{Mat}_k(\mathbb{C})$

$\mathrm{End}(V) = gl(V)$ Lie algebra $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$

Let $T(\mathrm{End}\, V) = T(gl(V))$

Let $\langle u\otimes v - v\otimes u - [u,v]\rangle$ be the ideal in $T(gl(V))$ generated by $u\otimes v - v\otimes u - [u,v]$, $u,v \in gl(V)$.

The universal enveloping algebra

$$U(gl(V)) \overset{\mathrm{def}}{=} T(gl(V)) \big/ \langle u\otimes v - v\otimes u - [u,v]\rangle$$

infinite-dimensional associative algebra

Example. $V = \mathbb{C}^2$, gl_2 2×2 matrices

Basis: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Relations: $[e,f] = ef - fe = h$ $\qquad [x, I] = 0 \ \forall x \in \{e,f,h\}$.

$[h,e] = he - eh = 2e$

$[h,f] = hf - fh = -2f$

$\Rightarrow U(gl_2) \simeq \mathbb{C}\langle e,f,h,I\rangle / \langle \text{Relations} \rangle$

$U(gl_2)$ contains $f^{\otimes 2} \overset{\text{notation}}{=} f^2$ $\qquad \sqcup fe \sqcup - \sqcup ef \sqcup = \sqcup h \sqcup$

$gl = \mathrm{End}(V)$ acts on V by $ev = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v$

$U(gl_2)$ acts on $V^{\otimes n}$ by :

$\Delta^k X = X \otimes 1 \otimes 1 \ldots \otimes 1 + 1 \otimes X \otimes 1 \otimes \ldots + \ldots + 1 \otimes 1 \otimes 1 \ldots \otimes X$ $\quad , \quad X \in gl(V)$.

$\Delta 1 = 1 \otimes 1$

$\Delta X = X \otimes 1 + 1 \otimes X$ has the property $\Delta(ab) = \Delta a \cdot \Delta b$, $a, b \in gl(V)$.

Check: $\Delta(ef - fe) = (e \otimes 1 + 1 \otimes e)(f \otimes 1 + 1 \otimes f) - (f \otimes 1 + 1 \otimes f)(e \otimes 1 + 1 \otimes e) =$

$= (ef - fe) \otimes 1 + 1 \otimes (ef - fe) = h \otimes 1 + 1 \otimes h = \Delta(h)$

$\Rightarrow \Delta : U(gl) \to U(gl) \otimes U(gl)$ defines a representation of $U(gl)$ in $V^{\otimes 2}$

$$\Delta^3 X = (id \otimes \Delta)\Delta X = (id \otimes \Delta)(1 \otimes X + X \otimes 1) = 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 + X \otimes 1 \otimes 1$$

$$\overset{\shortparallel}{(\Delta \otimes id)\Delta X}$$

$\Rightarrow$ We have the action of $U(gl(V))$ on $V^{\otimes n}$ for $n \in \mathbb{N}_+$

Theorem. Let $E = V^{\otimes n}$, V a fin dim vector space over $\mathbb{C}$.

Let $g \in S_n$ act by permutation of factors in $V^{\otimes n}$

$f : \mathbb{C}[S_n] \to End\, E$. Let $A = Im f \subset End\, E$

Then $B = End_A E$ is the image of $U(gl(V))$ in $End\, E$.

Proof. Let $b \in U(gl(V))$, clearly $\Delta^n b : V^{\otimes n} \to V^{\otimes n}$ commutes with the permutations of factors.

$$v \otimes w \xrightarrow{\Delta b} bv \otimes w + v \otimes bw \xrightarrow{(12)} w \otimes bv + bw \otimes v \qquad b \in gl(V)$$

$$\overset{(12)}{\searrow}\ w \otimes v \xrightarrow{\Delta b} bw \otimes v + w \otimes bv$$

(Since $\Delta^n X = X \otimes 1 \otimes .. \otimes 1 + 1 \otimes X \otimes 1 \otimes .. \otimes 1 + ... + 1 \otimes 1 \otimes .. \otimes X$ is symmetric).

$\Rightarrow \operatorname{Im}(U(gl(V)) \subset \operatorname{End}_A E = B$

$B = \operatorname{End}_A (V^{\otimes n}) = \operatorname{End}_{\mathbb{C}[S_n]}(V^{\otimes n}) = S^n(\operatorname{End} V)$ symmetric endomorphisms

Lemma. (1) If U is a $\mathbb{C}$-vector space, then $S^n U$ is spanned by elements $u \otimes ... \otimes u$, $u \in U$

(2) For a $\mathbb{C}$-algebra A, the algebra $S^n A$ is generated by the elements $\Delta^n(a) = a \otimes 1 \otimes .. \otimes 1 + 1 \otimes a \otimes . \otimes 1 + 1 \otimes ... \otimes a$, $a \in A$.

<u>Proof.</u> (1) $S^n\mathcal{U}$ is an irreducible representation of $GL(\mathcal{U})$ (PS 8, Ex. 4)

action : $g(u_1 \otimes u_2 \ldots \otimes u_k) = gu_1 \otimes gu_2 \otimes \ldots \otimes gu_k$

$g \in GL(\mathcal{U})$

$\Rightarrow g(u \otimes \ldots \otimes u) = gu \otimes \ldots \otimes gu \Rightarrow \text{Span}\{v \otimes \ldots \otimes v\} \subset S^n\mathcal{U}$ is a subrepresentation.

It is nonzero $\Rightarrow$ the whole $S^n\mathcal{U}$.

(2) Theorem on symmetric functions (Newton's identities):

$\exists$ polynomial with $\mathbb{Q}$ coefficients such that

$P(H_1(x), \ldots H_n(x)) = x_1 \ldots x_n$, where $H_i(x) = x_1^i + x_2^i + \ldots + x_n^i$

$$P((x_1 + \ldots + x_n), (x_1^2 + \ldots + x_n^2) \ldots (x_1^n + \ldots x_n^n)) = x_1 \ldots x_n$$

<u>Ex</u> $P((x_1+x_2), (x_1^2+x_2^2)) = -\frac{1}{2}(x_1^2+x_2^2) + \frac{1}{2}(x_1+x_2)^2 = x_1 x_2$

Then $P(\Delta^n(a), \Delta^n(a^2) \ldots \Delta^n(a^n)) = a \otimes \ldots \otimes a$ - span $\overset{\text{by (1)}}{S^n A}$.

$$x_i = 1 \otimes \ldots \otimes \underset{i}{a} \otimes \ldots \otimes 1 \Rightarrow x_1 x_2 \ldots x_n = a \otimes \ldots \otimes a$$

$\blacksquare$

Theorem (Schur-Weyl duality for $U(\mathfrak{gl}(V))$.)

(1) The image of A of $\mathbb{C}[S_n]$ and the image B of $U(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ are centralizers of each other

(2) Both A and B are semisimple, $V^{\otimes n}$ is a semisimple representation of $\mathbb{C}[S_n]$ and $U(\mathfrak{gl}(V))$.

(3) $V^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes L_\lambda$, λ partitions of n

$\{V_\lambda\}_\lambda$ are Specht modules for S_n, L_λ are inequivalent irreducible representations of $U(\mathfrak{gl}(V))$, or zero.

Proof $A = f(\mathbb{C}[S_n])$ is semisimple, and $B = \mathrm{End}_A E \Rightarrow$ by the double centralizer theorem, all statements follow. $\square$

Theorem. Schur-Weyl duality for $GL(V)$.

Let $GL(V)$ be the group of invertible matrices acting in V.

Then as a representation of $S_n \times GL(V)$ in $V^{\otimes n}$ decomposes as

$$V^{\otimes n} = \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda,$$ where V_λ are Specht modules

and $L_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ are distinct irreducible representations of S_n, or zero.

Examples. (1) $V = \mathbb{C}^k$, $E = V \otimes V$, $\mathbb{C}[S_2]$ acts on $V^{\otimes 2}$

$$V^{\otimes 2} \simeq S^2 V \oplus \Lambda^2 V \simeq V_{(2)} \otimes L_{(2)} \oplus V_{(1,1)} \otimes L_{(1,1)}$$

$$k^2 = \frac{(k+1)k}{2} + \frac{k(k-1)}{2}$$

$V_{(2)} = V_{triv}$, $V_{(1,1)} = V_{sign}$

(2) $V = \mathbb{C}^k$, $E = V^{\otimes n}$

$$\Rightarrow V^{\otimes n} \simeq V_{triv} \otimes S^n V \oplus \dots \oplus V_{sign} \otimes \Lambda^n V \qquad n > k$$

(3) $V = \mathbb{C}^2 \Rightarrow V^{\otimes 2} = V_{triv} \otimes L_2 \oplus V_{sign} \otimes L_0$

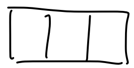
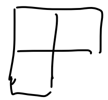
$\{e_1, e_2\} \subset V$

$L_2 \simeq \left\{ \begin{matrix} e_1 \otimes e_1,\ e_2 \otimes e_2, \\ e_1 \otimes e_2 + e_2 \otimes e_1 \end{matrix} \right\}$ $\quad \{e_1 \otimes e_2 - e_2 \otimes e_1\} \simeq L_0$

$\dim L_2 = 3$ $\quad \dim L_0 = 1$

$V = \mathbb{C}^2, \quad V^{\otimes 3} = V_{(3)} \otimes L_{(3)} \oplus V_{(2,1)} \otimes L_{(2,1)} \quad \ldots$

$\dim = 8$ $\quad 1 \quad 4 \quad 2$

conclusion: $\dim = 2$

$L_{(3)} = S^3 V \Rightarrow \dim L_{(3)} = \binom{3+2-1}{3} = 4$

Only the partitions with # of rows $\leq \dim V$ will appear.

— The End —

Representation theory

Math-314 $\boxed{G \text{ finite over } \mathbb{C}}$

Schur-Weyl

Geometry of $G, \mathfrak{g}$

Compact Lie groups over $\mathbb{C}$ $\simeq$ Semisimple complex Lie alg.

Geometry of $\hat{G}, \hat{\mathfrak{g}}$

Affine Kac-Moody algebras

Double affine alg.

equivalences

Quantum affine alg.

Quantum groups q

Elliptic q-gps q, t

G finite over K, char $K > 0$

Schur-Weyl

Algebraic groups over K, char $K > 0$

equivalences

Quantum groups with $q^p = 1$