Lecture 13

Recall: representations of S_n

(1) $V_\lambda = \mathbb{C}[S_n]c_\lambda \quad \longleftrightarrow \quad \lambda$ partitions of n irreducible

$V_\lambda \simeq V_\mu \iff \lambda = \mu$.

(2) $\mathcal{U}_\lambda = \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{triv} \qquad \lambda = (\lambda_1, \lambda_2 \dots \lambda_p) \qquad \mathcal{U}_\lambda = \mathbb{C}[S_n]a_\lambda$

Character: $\chi_{\mathcal{U}_\lambda}(C_{\underline{i}}) = \text{coeff of } x^\lambda = \prod_j x_j^{\lambda_j}$ in $\prod_{m \geq 1} H_m(x)^{i_m}$, where

$C_{\underline{i}}$ is of type $(i_1\, i_2 \dots i_\ell \dots)$ and $H_m(x) = (x_1^m + \dots + x_N^m)$, $N \geq p$.

Today: (1) Character of V_λ

(2) The hook length formula for the dimension of V_λ

Consider $\Delta(x) = \prod_{1 \le i < j \le N} (x_i - x_j)$ Let $\rho = (\underbrace{N-1, N-2, \ldots\ldots 1, 0}_{\sigma \text{ permutes}})$

Then $\Delta(x) = \sum_{\sigma \in S_N} (-1)^{\sigma} x^{\sigma(\rho)} = x_1^{N-1} x_2^{N-2} x_{N-1} + \ldots$

$x^{\rho} \stackrel{def}{=} x_1^{N-1} x_2^{N-2} \ldots x_{N-1}$; $x^{\sigma(\rho)} = \prod_i x_i^{\sigma(\rho_i)}$

$$\Delta(x) = \text{Vandermonde determinant} = \det \begin{pmatrix} x_1^{N-1} & x_1^{N-2} & \ldots & x_1 & 1 \\ x_2^{N-1} & x_2^{N-2} & & x_2 & 1 \\ & & & & \\ x_N^{N-1} & x_N^{N-2} & & x_N & 1 \end{pmatrix} \doteq \prod_{i<j} (x_i - x_j)$$

Ex. Vandermonde $N=3$

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2 x_2 + x_3^2 x_1 + x_2^2 x_3 - x_3^2 x_2 - x_2^2 x_1 - x_1^2 x_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

Theorem. $N \geq p$, $\lambda = (\lambda_1 \dots \lambda_p)$ $\quad C_{\bar{i}} : \bar{i} = (i_1\ i_2\ i_e \dots)$

$\chi_{V_\lambda}(C_i)$ is the coefficient Θ_λ of $X^{\lambda+\rho} = \prod_j x_j^{\lambda_j + N - j}$

in the polynomial $\Delta(X) \prod_{m \geq 1} H_m(X)^{i_m}$, $\Delta X = \prod_{1 \leq i < j \leq N} (X_i - X_j)$.

Proof (1) Θ_λ is a class function : depends on λ and the type of $C_{\bar{i}}$

Coeff of $X^{\lambda+\rho}$ in $X_1^{N-1} X_2^{N-2} \dots X_{N-1} \prod_{m \geq 1} H_m(X)^{i_m}$

$X^\lambda \cdot (X_1^{N-1} X_2^{N-2} \dots X_{N-1})$ in $(X_1^{N-1} X_2^{N-2} \dots X_{N-1}) \prod_{m \geq 1} H_m(X)^{i_m}$ equals

to the coeff of X^λ in $\prod_{m \geq 1} H_m(X)^{i_m} = \chi_{U_\lambda}(C_{\bar{i}})$

coeff of $X^{\lambda+\rho}$ in $X^{\sigma(\rho)} \prod_{m \geq 1} H_m(X)^{i_m}$ equals to

coeff $X^{\lambda + \sigma(\rho) + \rho - \sigma(\rho)}$ in $X^{\sigma(\rho)} \prod_{m \geq 1} H_m(X)^{i_m}$ equals to

$\parallel$

coeff $X^{(\lambda + \rho - \sigma(\rho)) + \sigma(\rho)}$ in $X^{\sigma(\rho)} \prod_{m \geq 1} H_m(X)^{i_m} = \chi_{U_{\lambda+\rho-\sigma(\rho)}}$

(dropped if negative entries appear in $\lambda + \rho - \sigma(\rho)$)

$$\Rightarrow \quad \Theta_\lambda = \sum_{\sigma \in S_N} (-1)^\sigma \chi_{U_{\lambda+\rho-\sigma(\rho)}}(C_i) = \sum_{\mu \geq \lambda} (-1)^{\sigma(\mu)} \chi_{U_\mu}(C_i) =$$

$$\lambda+\rho-\sigma(\rho) \geq \lambda \qquad \lambda+\rho-\sigma(\rho) = \lambda \Leftrightarrow \sigma = 1$$

$$= \sum_{\mu \geq \lambda} L_{\lambda\mu} \chi_{U_\mu}(C_i) \qquad L_{\lambda\lambda} = 1.$$

Recall the Kostka numbers decomposition: $\chi_{U_\lambda} = \sum_{\mu \geq \lambda} K_{\lambda\mu} \chi_{V_\mu}$

s.t. $K_{\lambda\lambda} = 1$.

$$\Rightarrow \quad \Theta_\lambda = \sum_{\mu \geq \lambda} M_{\lambda\mu} \chi_{V_\mu}, \quad M_{\lambda\lambda} = 1$$

$$= \chi_{V_\lambda} + \sum_{\mu > \lambda} M_{\lambda\mu} \chi_{V_\mu}$$

(2) It suffices to show that $(\Theta_\lambda, \Theta_\lambda) = 1$.

See Etingof.

$\blacksquare$

Question: dimension of V_λ ?

$\dim V_\lambda = $ coeff of $x^{\lambda+\rho}$ in $\Delta(x)(x_1+\dots+x_N)^n$

$C_i = 1 = n$ 1-cycles

Theorem $\dim V_\lambda = \dfrac{n!}{\prod_j \ell_j!} \prod_{1\le i<j\le N} (\ell_i - \ell_j)$

where $\ell_j \overset{def}{=} \lambda_j + N - j$ powers of x_j in $x^{\lambda+\rho}$

Proof. Coeff of $x^{\lambda+\rho}$ in $x^\rho(x_1+\dots+x_N)^n$ is $\dfrac{n!}{\lambda_1!\lambda_2!\dots\lambda_p!}$

Coeff of $x^{\lambda+\rho}$ in $\underbrace{x^{\sigma(\rho)}}\underbrace{(x_1+\dots+x_N)^n}$

Need: $x_j^{\lambda_j+N-j} = x_j^{\ell_j}$. have : $x_j^{N-\sigma(j)}$ remains to take: $x_j^{\ell_j - N+\sigma(j)}$ if $\ell_j \ge N-\sigma(j)$

$\Rightarrow$ Coeff of $x^{\lambda+\rho}$ in $x^{\sigma(\rho)}(x_1+\dots+x_N)^n$ is $\dfrac{n!}{\prod_j(\ell_j - N+\sigma(j))!}$

$\Rightarrow$ coeff of $x^{\lambda+\rho}$ in $\Delta(x)(x_1+\dots+x_N)^n$ is

$$\Rightarrow \dim V_\lambda = \sum_{\substack{\sigma\in S_N \\ l_i \geq N-\sigma(i)}} (-1)^\sigma \frac{n!}{\prod_i (l_i - N+\sigma(i))!} =$$

$$= \frac{n!}{l_1!\dots l_N!} \sum_{\substack{\sigma\in S_N \\ l_i\geq N-\sigma(i)}} (-1)^\sigma \frac{l_1!\dots l_N!}{\prod_i (l_i-N+\sigma(i))!} = \frac{n!}{\prod_j l_j!} \sum_{\sigma\in S_N} (-1)^\sigma \prod_j l_j(l_j-1)\dots(l_j-N+\sigma(j)+1) =$$

$\underline{\text{if } l_i < N-\sigma(i)} \Rightarrow l_i - N+\sigma(i) < 0$

$l_i - N+\sigma(i)+1 \leq 0$

$\Rightarrow$ the product contains 0.

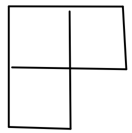
$$= \frac{n!}{\prod_j l_j!} \underbrace{\sum_{\sigma\in S_N} (-1)^\sigma \prod_j a_{j\sigma(j)}}_{\text{" } \det a_{ij}} = \frac{n!}{\prod_j l_j!} \det\Big(\underbrace{l_j(l_j-1)\dots(l_j-N+i+1)}_{a_{ij}}\Big)$$

$$\det \begin{vmatrix} \ell_1(\ell_1-1)\cdots(\ell_1-N+2) & \ell_1(\ell_1-1)\cdots(\ell_1-N+3) & & \ell_1(\ell_1-1) & \ell_1 & 1 \\ \ell_2(\ell_2-1)\,(\ell_2-N+2) & \ell_2(\ell_2-1)\,(\ell_2-N+3 & & \ell_2(\ell_2-1) & \ell_2 & 1 \\ & & & & & \\ \ell_N(\ell_N-1) & & & \ell_N(\ell_N-1) & \ell_N & 1 \end{vmatrix} =$$

$$= \det \begin{vmatrix} \ell_1^{N-1} & \ell_1^{N-2} & \cdots & \ell_1 & 1 \\ \ell_2^{N-1} & & & & 1 \\ & & & & \\ \ell_N^{N-1} & & & \ell_N & 1 \end{vmatrix} = \text{Vandermonde} = \prod_{1\le i<j\le N} (\ell_i-\ell_j)$$

$$\Rightarrow \dim V_\lambda = \frac{n!}{\prod_j \ell_j!} \prod_{1\le i<j\le N} (\ell_i-\ell_j) \qquad \text{where } \ell_i = \lambda_i + N - i$$

$\blacksquare$

Example. S_3 $\quad \lambda = (2,1,0)$ $\quad$ 

$N=3$

$$\ell_1 = 2+3-1 = 4$$
$$\ell_2 = 1+3-2 = 2$$
$$\ell_3 = 0+3-3 = 0$$

$$\dim V_{(2,1)} = \frac{3!}{4!\,2!\,0!}(4-2)(4-0)(2-0) = $$
$$= \frac{1}{4\cdot 2}\, 2\cdot 2\cdot 4 = 2.$$

$$\frac{3!}{4!\,2!\,0!}\begin{vmatrix} \ell_1(\ell_1-1) & \ell_1 & 1 \\ \ell_2(\ell_2-1) & \ell_2 & 1 \\ \ell_3(\ell_3-1) & \ell_3 & 1 \end{vmatrix} = \frac{3!}{4!\,2!\,0!}\begin{vmatrix} 4\cdot 3 & 4 & 1 \\ 2\cdot 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{4\cdot 2}\cdot 16 = 2.$$

$$\dim V_{(2,1)} = \text{coeff of } x^{\lambda+\rho} = x_1^{2+3-1}\, x_2^{1+3-2}\, x_3^{0+0} = x_1^4 x_2^2 \text{ in } \Delta(x)(x_1+x_2+x_3)^3$$

$$(\underbrace{x_1^2x_2}_{\sigma=1} + x_2^2x_3 + x_3^2x_1 - x_3^3x_2 - x_1^2x_3 - \underbrace{x_2^2x_1}_{\sigma=(12)})(x_1^3 + x_2^3 + x_3^3 + 3(x_1^2x_2 + x_2^2x_3 + x_3^2x_2 + x_3^2x_1 + x_1^2x_3 + x_2^2x_1) +$$
$$+ 6x_1x_2x_3) = 3-1 = 2.$$

$$\sum (-1)^\sigma x^{\sigma(\rho)}$$

$\sigma(3)=3$ for $\ell_3 \geq N-\sigma(3)$

$0 \geq 3-\sigma(3) \Rightarrow \sigma(3)=3 \Rightarrow$ the only contributions from $\sigma=1$, $\sigma=(12)$

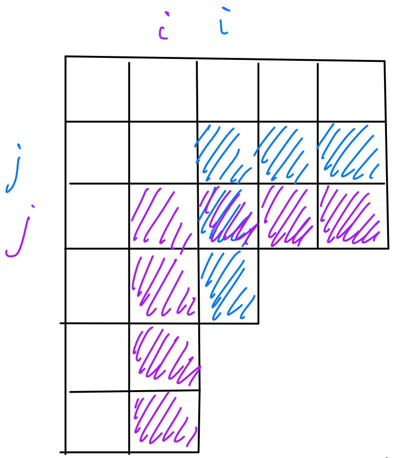
Exercise $\dim V_{(2,1)}$ with $N=2$ (easier)

Hook length formula.

Def Let Y_λ be a Young diagram. $i \in \lambda_j$, $(i\,j)$ denotes the i-th j-th square.

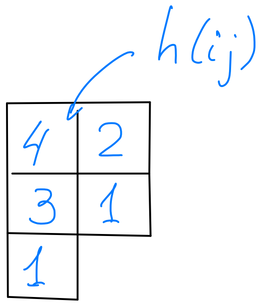
Then $h(i,j) \stackrel{\text{def}}{=}$ is the number of squares in the hook in Y_λ starting at (i,j) and going to the right and down in Y_λ.

Ex



$h(ij) = 5$

$h(ij) = 7$



Theorem.

$$\dim V_\lambda = \frac{n!}{\prod_{i \in \lambda_j} h(i,j)}$$

$$\dim V_\lambda = \frac{n!}{\prod_j l_j!} \prod_{1\le i<j\le N} (l_i - l_j) \quad \text{where } l_i = \lambda_i + N - i \ ; \ \text{let } N = p$$

$$\text{Consider } \frac{l_1!}{\prod_{1<j\le p} (l_1 - l_j)} = \frac{l_1!}{(l_1-l_2)\cdots(l_1-l_p)} = \prod_{i\in\lambda_1} h(i,1)$$

$l_1 = \lambda_1 + p - 1$ = # squares in (1st row + 1st column) $- 1 = h(1,1)$

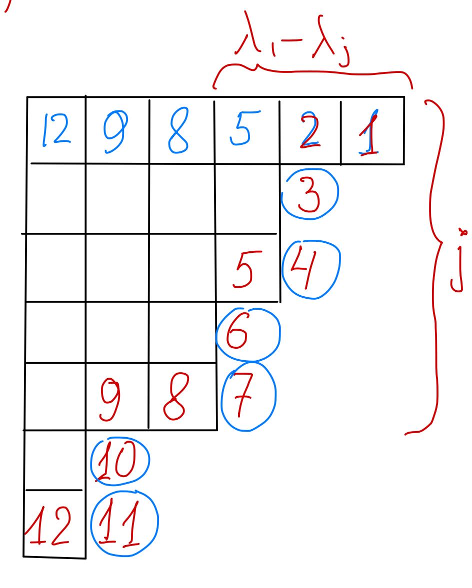
$$l_1 - l_j = \lambda_1 + p - 1 - \lambda_j - p + j =$$
$$= \underline{\lambda_1 - \lambda_j + j - 1}$$

$\lambda_1 - \lambda_2 + 2 - 1$

$\lambda_1 - \lambda_j + j - 1$

$\lambda_1 - \lambda_{p-1} + p - 2$

$\lambda_1 - \lambda_p + p - 1$



$\Rightarrow l_1 - l_j = \lambda_1 - \lambda_j + j - 1$ = j-th number to skip

Put $l_1 = h(1,1)$ in the lower left corner of Y_λ. Then go down from l_1 by one, putting $(k-1)$ to the right of k if $k \in Y_\lambda$, or $(k-1)$ above k if $k \notin Y_\lambda$. The numbers outside of Y_λ are exactly the numbers to skip in $l_1!$ to get $\prod_{i\in\lambda_1} h(i,1)$

In j-th row $\ell_1-\ell_j = \lambda_1-\lambda_j+j-1$ = circled number outside of Y_λ = number to skip.

$$= (\lambda_1-\lambda_j) + j - 1$$

by how many squares λ_1 is longer than λ_j

how many squares down from 1st to j-th row

not counting the corner twice

Similarly for other rows $\quad \dfrac{\ell_i!}{\prod_{i<j\le p}(\ell_i-\ell_j)} = \prod_{k\in\lambda_i} h(k\,i)$

$$\Rightarrow \text{Totally } \dim V_\lambda = \frac{n!}{\prod_j \ell_j!}\prod_{1\le i<j\le N}(\ell_i-\ell_j) = \frac{n!}{\prod_{i\in\lambda_j} h(ij)} \qquad \blacksquare$$

Ex. $\quad S_3$.

$$\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 3 & 1\\\hline 1 & \multicolumn{1}{c}{}\\\cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1\\\hline\end{array}$$

$\dim_{V_{(1,1,1)}} = \dfrac{3!}{3\cdot 2\cdot 1} = 1 \qquad \dim_{V_{(2,1)}} = \dfrac{3!}{3\cdot 1} = 2 \qquad \dim_{V_{(3)}} = \dfrac{3!}{3!} = 1$