Lecture 12

Recall: representations of S_n.

Let λ be a partition of n, Y_λ the Young diagram, T_λ a Young tableau.

Let P_λ = group of permutations along the rows of T_λ, $a_\lambda = \sum_{g \in P_\lambda} g$

Q_λ = group of permutations along the columns of T_λ, $b_\lambda = \sum_{g \in Q_\lambda} (-1)^g g$.

Let $c_\lambda = a_\lambda b_\lambda$.

Theorem. Let $V_\lambda \overset{def}{=} \mathbb{C}[S_n] c_\lambda$ the Specht module $\leftrightarrow \lambda$.

Then (1) V_λ is irreducible

(2) Every irreducible S_n-module is isomorphic to V_λ for a unique $\lambda \vdash n$.

Lemma 1 Let $x \in \mathbb{C}[S_n]$. Then $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, $\ell_\lambda(x) \in \mathbb{C}$

Today: (1) Finish classification of the irreducible representations of S_n.

(2) $V_\lambda \subset U_\lambda = \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{triv}$; characters of S_n-representations.

Def. Lexicographic order in partitions of n: $\lambda > \mu$ if the first nonzero $\lambda_i - \mu_i > 0$

Ex. $(n) >$ any other partition of n; $(3,2) > (3,1,1)$

$(5,4,1,1) > (5,3,2,1)$

Lemma 2 If $\lambda > \mu$, then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$

Proof: Need to show $\forall g \in S_n$ $\exists t \in P_\lambda$ (transposition) s.t. $g^{-1} t g \in Q_\mu$

Then $a_\lambda g b_\mu = a_\lambda t g b_\mu = a_\lambda g (g^{-1} t g) b_\mu = -a_\lambda g b_\mu = 0$.

There exist i, j in the same row of T_λ and the same column of $g T_\mu$.

if $\lambda_1 > \mu_1 \Rightarrow$ T_λ

 λ_1

$g T_\mu$

 μ_1

max # of columns $= \mu_1 < \lambda_1$

$\Rightarrow$ by the pigeonhole principle $\exists$ a column that contains two elts from the 1st row of T_λ.

If $\lambda_1 = \mu_1 \Rightarrow p_1 \in P_\lambda$, $q_1' \in g Q_\mu g^{-1}$ s.t. $(p_1 T_\lambda)_{1st\,row} = (q_1' T_\mu)_{1st\,row}$

Eventually since $\lambda > \mu \Rightarrow \exists\, \lambda_i > \mu_i$ and there are two elts in the λ_i row of T_λ that are in the same column of $g T_\mu$.

Our operations only permuted elts in rows of T_λ and in columns of $g T_\mu \Rightarrow$ does not change the existence of i, j in the same row of T_λ and same column of $g T_\mu$.

$\Rightarrow \exists\, (ij) \in P_\lambda$, $g^{-1}(ij)g \in Q_\mu$. $\blacksquare$

Ex. S_3 $\quad \underset{\sigma}{(3)} > \underset{\lambda}{(2,1)} > \underset{\mu}{(1,1,1)}$ $\qquad \underset{\sigma}{\boxed{1\,|\,2\,|\,3}} \quad \underset{\lambda}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}} \quad \underset{\mu}{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}}$

$$a_\sigma = \frac{1}{6}\sum_{g\in S_3} g \quad ; \quad a_\lambda = \frac{1}{2}\left(1+(12)\right) \qquad b_\mu = \sum_{g\in S_3}(-1)^g g$$

$$b_\lambda = \frac{1}{2}\left(1-(13)\right)$$

$$a_\sigma b_\lambda = \frac{1}{6}\sum g \cdot \frac{1}{2}\left(1-(13)\right) = \frac{1}{12}\sum_g \left(g-(13)g\right) = 0$$

$$a_\lambda b_\mu = \frac{1}{12}\sum(-1)^g\left(g+(12)g\right) = 0$$

$$1 \quad + (12)$$

$$-(12) - 1$$

Lemma 3. $c_\lambda^2 = \alpha\, c_\lambda$ $\quad , \alpha\in\mathbb{C}$ $\quad$, α depends on λ.

PS 12: find α.

$$\Rightarrow c_\lambda^2 = \alpha\, c_\lambda \Rightarrow \left(\frac{c_\lambda}{\alpha}\right)^2 = \left(\frac{c_\lambda}{\alpha}\right) \quad ; \quad \text{so } \left(\frac{c_\lambda}{\alpha}\right) \text{ is an idempotent}$$

Lemma 4. Let A be an associative algebra and $e \in A$: $e^2 = e$ in A. Let M be a left A-module

Then $\mathrm{Hom}_A(\underbrace{Ae}_{\text{left } A\text{-mod}}, M) \simeq eM$

Proof. Consider first $\mathrm{Hom}_A(A, M) \underset{\beta\,\rightarrow}{\overset{\alpha\,\leftarrow}{\simeq}} M$

$f \in \mathrm{Hom}_A(A, M)$

$\Rightarrow f(a) = v \in M$; $f(xa) = x(f(a)) = xv$ since $f \in \mathrm{Hom}_A$

Define $\underbrace{\alpha(v)}_{f}(a) \overset{def}{=} av \in M$

Then $\alpha(v)(xa) = xav = x(\alpha(v)(a)) \Rightarrow \alpha(v) \in \mathrm{Hom}_A$

Define $\beta(f) \overset{def}{=} f(1)$

Then $\beta \circ \alpha(v) = \alpha(v)(1) = 1 \cdot v = v \Rightarrow \beta\circ\alpha = Id_M$

$\alpha \circ \beta(f)(a) = a\beta(f) = a f(1) = f(a) \Rightarrow \alpha\circ\beta = Id_{Hom_A(A,M)}$

$f \in Hom_A$

$\Rightarrow Hom_A(A, M) \simeq M$

Let $e \in A$: $e^2 = e \Rightarrow (1-e)^2 = 1 - 2e + e^2 = 1 - e$

For any vector $v \in M$; $v = ev + (1-e)v$

$\Rightarrow M = eM \oplus (1-e)M$

Consider $\alpha(eM)$: $\alpha(ev)(a) = aev = ae \cdot ev = \alpha(ev)(ae)$

$\Rightarrow \alpha(ev) \in Hom_A(Ae, M)$

$\alpha(ev)(a(1-e)) = a(1-e)ev = a(e^2 - e)v = 0$

$\Rightarrow Hom_A(Ae + A(1-e), M) \simeq eM \oplus (1-e)M$

where $Hom_A(Ae, M) \simeq eM$

$\blacksquare$

Theorem. $V_\lambda = \mathbb{C}[S_n]c_\lambda$ are irreducible, and $\{V_\lambda\}_{\lambda \text{ partitions of } n}$ form a complete set of inequivalent irreducible $\mathbb{C}[S_n]$-modules.

Proof. Let $\lambda > \mu$.

$$\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = \mathrm{Hom}_{S_n}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu) \simeq c_\lambda \mathbb{C}[S_n]c_\mu$$

$$\mathrm{Hom}_A(Ae, M) \simeq eM$$

We have by Lemma 2 $\quad c_\lambda \mathbb{C}[S_n]c_\mu = a_\lambda b_\lambda \mathbb{C}[S_n]a_\mu b_\mu = 0$ if $\lambda > \mu$

by Lemma 1 $\quad c_\lambda \mathbb{C}[S_n]c_\lambda = a_\lambda b_\lambda \mathbb{C}[S_n]a_\lambda b_\lambda =$

$= a_\lambda \mathbb{C}\text{eff}\, b_\lambda$

is 1-dimensional.

$$V_\lambda = \bigoplus_i \underset{\text{irreducibles}}{V_i}^{n_i} \Rightarrow \dim \mathrm{Hom}_{S_n}(V_\lambda, V_\lambda) = \sum n_i^2 = \underline{1}$$

$\Rightarrow V_\lambda$ is irreducible.

Since $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = 0$ for $\lambda > \mu \Rightarrow \underset{\lambda > \mu}{V_\lambda \not\simeq V_\mu}$

$\Rightarrow \{V_\lambda\}_{\lambda \text{ partitions of } n}$ are pairwise non-isomorphic.

$\#$ conjugacy classes in S_n = $\#$ partitions of n = $\# V_\lambda$. $\blacksquare$

<u>Remark:</u> $V_\lambda = \mathbb{C}[S_n]c_\lambda$, c_λ is defined using P_λ, Q_λ

For different T_λ, get different c_λ

However, we proved that $\mathbb{C}[S_n]c_\lambda \simeq \mathbb{C}[S_n]c'_\lambda \simeq V_\lambda$.

Induced representations of S_n

$$a_\lambda = \sum_{g \in P_\lambda} g$$

Let $U_\lambda = \mathbb{C}[S_n]a_\lambda \simeq \mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{triv}$

$$\mathrm{Ind}_{P_\lambda}^{S_n} \mathbb{C}_{triv} \simeq \mathbb{C}[S_n] \otimes_{\mathbb{C}[P_\lambda]} \mathbb{C}_{triv} \simeq \mathbb{C}[S_n]a_\lambda$$

$$x \otimes 1 \longrightarrow x a_\lambda$$

$$\underset{h\in P_\lambda}{xh} \otimes 1 = x \otimes 1 \longrightarrow x h a_\lambda = x a_\lambda \quad h \in P_\lambda$$

Proposition. $\mathrm{Hom}(U_\lambda, V_\mu) = 0$ for $\lambda > \mu$

$\dim \mathrm{Hom}(U_\lambda, V_\lambda) = 1.$

Proof: $\mathrm{Hom}(U_\lambda, V_\mu) = \mathrm{Hom}(\mathbb{C}[S_n]a_\lambda, \mathbb{C}[S_n]a_\mu b_\mu) \simeq a_\lambda \mathbb{C}[S_n]a_\mu b_\mu = 0, \lambda > \mu$

1-dim, $\mu = \lambda$

$\blacksquare$

$$\Rightarrow \mathcal{U}_\lambda = \bigoplus_{\mu \geq \lambda} K_{\mu\lambda} V_\mu \quad \text{and} \quad K_{\lambda\lambda} = 1.$$

$\{K_{\mu\lambda}\}$ are the Kostka numbers.

Characters of representations of S_n.

We will start with the character of $\mathcal{U}_\lambda$.

Theorem. Let $C_{\bar{i}}$ be the conjugacy class in S_n having i_ℓ cycles of length ℓ $\quad \bar{i} = (i_1, \dots i_\ell \dots)$

$$g \in C_{\bar{i}} \;:\; g = (\underbrace{(\,)(\,)(\,)}_{i_1}\underbrace{(\cdot\cdot)(\cdot\cdot)}_{i_2} \dots \underbrace{(\overset{\ell}{\cdots})(\overset{\ell}{\cdots})}_{i_\ell} \dots)$$

Let $\lambda = (\lambda_1, \dots \lambda_p)$ be a partition of n.

Let $N \geq p$, $\{x_1, \dots x_N\}$ variables. Let $H_m(x) = (x_1^m + x_2^m + \dots + x_N^m)$

Then $\chi_{\mathcal{U}_\lambda}(C_{\bar{i}})$ equals the coefficient of $x^\lambda = \prod_j x_j^{\lambda_j}$ in the polynomial

$$\prod_{m \geq 1} H_m(x)^{i_m}$$

Proof. Character of an induced representation,

$$\chi_{U_\lambda}(C_i) = \frac{1}{|P_\lambda|} \sum_{x \in G : xgx^{-1} \in P_\lambda} \chi_{V_0}(xgx^{-1}) =$$
$g \in C_i$

$$= \frac{1}{|P_\lambda|} \left(\# \{ x \in S_n : xgx^{-1} \in P_\lambda, g \in C_i \} \right) \quad ; \quad |P_\lambda| = \prod_j \lambda_j !$$

$$\# \, x : xgx^{-1} \in P_\lambda = |C_i \cap P_\lambda| \cdot |Z_g|$$

Centralizer of an elt in $C_i \ni g$

$$Z_g \simeq \prod_m S_{i_m} \ltimes (C_m)^{i_m} \Rightarrow |Z_g| = \prod_m (i_m)! \, m^{i_m}$$

swap disjoint cycles of same length

Next: Find $|C_i \cap P_\lambda|$.

$\#$ of elts with r_{mj} m-cycles in S_{λ_j} $= \dfrac{\lambda_j!}{\prod_{m\geq 1} m^{r_{jm}}(r_{jm})!}$

$\boxed{\sum_m m r_{jm} = \lambda_j}$; $\boxed{\sum_j r_{jm} = i_m}$

As above, we can compute $\prod_{m\geq 1} m^{r_{jm}}(r_{jm})! = |Z(r_{mj}$ cycles of length $m)|$ in S_{λ_j}

$$\Rightarrow |C_i \cap P_\lambda| = \sum_r \prod_{j\geq 1} \frac{\lambda_j!}{\prod_{m\geq 1} m^{r_{jm}}\, r_{jm}!}$$

$\prod_{j\geq 1} m^{r_{jm}} = m^{i_m}$

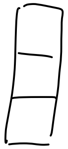
$$\Rightarrow \chi_{u_\lambda}(C_i) = \frac{|Z_g||C_i \cap P_\lambda|}{|P_\lambda|} = \sum_r \frac{\prod_m \cancel{m^{i_m}}(i_m)!}{\prod_j \cancel{\lambda_j!}} \prod_{j\geq 1} \frac{\cancel{\lambda_j!}}{\prod_m \cancel{m^{r_{jm}}}(r_{jm})!} =$$

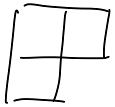
$$= \sum_r \prod_m \frac{i_m!}{\prod_j r_{jm}!}$$ this is exactly the coefficient

of $\prod_j x_j^{\lambda_j}$ in $\prod_{m\geq 1}(x_1^m + \dots + x_N^m)^{i_m}$, where we pick r_{mj} x_j^m from $H_m(x)^{i_m}$

Multinomial coefficient: $(x_1 + \dots + x_r)^n = \sum_{\sum k_i = n} \dfrac{n!}{k_1!\,k_2!\dots k_r!} x_1^{k_1} x_2^{k_2} \dots x^{k_r}$ $\blacksquare$

Example. S_3 $\quad \lambda = (3)$  $P_\lambda = S_3$ $\quad \mathcal{U}_{(3)} = \mathrm{Ind}_{S_3}^{S_3} \mathbb{C}_0 = \mathbb{C}_0$

$\lambda =$  $(1,1,1)$ $\Rightarrow P_\lambda = \{1\} \Rightarrow \mathcal{U}_{(1,1,1)} = \mathrm{Ind}_{\{1\}}^{S_3} \mathbb{C}_0 = \mathbb{C}[S_3]$ regular

$\lambda = (2,1)$  $\Rightarrow \mathcal{U}_{(2,1)} = \mathrm{Ind}_{C_2}^{S_3} \mathbb{C}_0 \simeq \mathbb{C}[S_3] \otimes_{\mathbb{C}[C_2]} \mathbb{C}_0 \simeq V_2 \oplus V_0$

$\simeq \mathbb{C}[S_3] a_\lambda$

exercise: compute this explicitly by the action on $a_\lambda = \frac{1}{2}(1 + (12))$.

Characters: $\lambda = (2,1)$; let $N = p = 2 \Rightarrow x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} = x_1^2 x_2$.

$(1) = C_i$ $\Rightarrow$ coeff $x_1^2 x_2$ in $(x_1 + x_2)^3 = x_1^3 + \underline{3} x_1^2 x_2 + 3 x_2 x_1^2 + x_2^3$
3 1-cycles

$\Rightarrow \chi_{(2,1)}(1) = 3$

$(12) = C_i$
1 2-cycle, 1 1-cycle $\Rightarrow$ coeff of $x_1^2 x_2$ in $(x_1^2 + x_2^2)(x_1 + x_2) = \underline{\ } x_1^2 x_2 + x_1^3 + x_2^2 x_1 + x_2^3$

$\Rightarrow \chi_{(2,1)}((12)) = \underline{1}$

$(123) = C_i$
1 3-cycle $\Rightarrow$ coeff of $x_1^2 x_2$ in $(x_1^3 + x_2^3)$ is zero

$\Rightarrow \chi_{(2,1)}(123) = 0$

Compare with the character table for S_3:

	1	(12)	(123)
V_0	1	1	1
V_S	1	-1	1
V_2	2	0	-1
$\mathcal{U}_{(12)}$	3	1	0

$$\Rightarrow \mathcal{U}_{(12)} \simeq \mathrm{Ind}_{P_{(12)}}^{S_3} V_{\mathrm{triv}} \simeq V_0 \oplus V_2$$