Lecture 11

Recall: Induced representation.

Def. Let $H < G$ and V a representation of H. The induced representation

$$\mathrm{Ind}_H^G V = \{ f : G \to V : f(hx) = \rho_V(h) f(x) \quad \forall x \in G, h \in H \}$$

Action of G: $\quad g \cdot f(x) = f(xg) \quad \forall g, x \in G$

Frobenius character formula: Let $x_\sigma \in H\backslash G$ representatives of right H-cosets.

Then $$\chi_{\mathrm{Ind}_H^G V}(g) = \sum_{\sigma \in H\backslash G \, : \, \sigma g = \sigma} \chi_V(x_\sigma g x_\sigma^{-1}).$$

Reciprocity: $\quad \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G V, W).$

Today: (1) More on $\mathrm{Ind}_H^G V$, $\mathrm{Res}_H^G V$.

(2) Irreducible representations of S_n.

Another viewpoint on Ind and Res.

$$k[H] \curvearrowright k[G] \curvearrowleft k[G] \qquad\qquad k[G] \curvearrowright k[G] \curvearrowleft k[H]$$

$k[G]_1$ - $k[H]$-$k[G]$-bimodule $\qquad$ $k[G]_2$ $k[G]$-$k[H]$ bimodule

If V is a right A-module and W is a left A-module,

then $\quad V \underset{A}{\otimes} W \simeq V \otimes W \big/ \langle va \otimes w - v \otimes aw \rangle_{a \in A}$

<u>Claim 1</u> $\quad \mathrm{Res}^G_H V = k[G]_1 \underset{k[G]}{\otimes} V \qquad \dim = \dim V$

$F : \mathrm{Res}^G_H V \to k[G]_1 \underset{k[G]}{\otimes} V \qquad F(v) = 1 \otimes v \quad$ injective

H-homomorphism: $F(hv) = 1 \underset{k[G]}{\otimes} hv = h \underset{k[G]}{\otimes} v = h(1 \otimes v) = h(F(v))$.

Claim 2 $\quad \operatorname{Ind}_H^G \bar{W} \simeq k[G]_2 \underset{k[H]}{\otimes} \bar{W}$ $\qquad \dim k[G]_2 \underset{k[H]}{\otimes} \bar{W} = \frac{|G|}{|H|} \dim \bar{W}$

$$T : \operatorname{Ind}_H^G \bar{W} \to k[G]_2 \underset{k[H]}{\otimes} \bar{W} \qquad T(f) = \sum_{x_\sigma} x_\sigma^{-1} \underset{k[H]}{\otimes} f(x_\sigma)$$

(1) $T(f)$ does not depend on the choice of representative

If $y_\sigma = h x_\sigma \Rightarrow f \to \sum_\sigma x_\sigma^{-1} \underset{k[H]}{\otimes} f(x_\sigma)$

$$\searrow \sum_\sigma y_\sigma^{-1} \underset{k[H]}{\otimes} f(y_\sigma) = \sum_\sigma (x_\sigma^{-1} h^{-1}) \underset{k[H]}{\otimes} f(h x_\sigma) =$$

$$= \sum_\sigma x_\sigma^{-1} \otimes h^{-1} h f(x_\sigma) = \sum_\sigma x_\sigma^{-1} \otimes f(x_\sigma).$$

(2) Maps basis to basis : a bijection.

$\operatorname{Ind}_H^G \bar{W} = \{ \delta_\sigma^i : \delta_\sigma^i (x_\mu) = \delta_{\sigma\mu} w^i \}$ where $\{w^i\}$ is a basis in $\bar{W}$.

$k[G]_2 \underset{k[H]}{\otimes} \bar{W} = \{ x_\sigma^{-1} \otimes w_i \}_{\sigma \in H \backslash G}$, $\{w^i\}$ a basis in $\bar{W}$

$$T(\delta_\sigma^i) = \sum_\mu x_\mu^{-1} \underset{k[H]}{\otimes} \delta_{\sigma_\mu}(x_\mu) w^i = x_\sigma^{-1} \otimes w^i$$

(3) T is an G-homomorphism:

$$g\, T(f) = \sum_\sigma g x_\sigma^{-1} \underset{k[H]}{\otimes} f(x_\sigma) = \sum_\sigma x_\mu^{-1} h \otimes f(h^{-1} x_\mu g) =$$

Let h: $g x_\sigma^{-1} = x_\mu^{-1} h \Rightarrow x_\sigma g^{-1} = h^{-1} x_\mu \Rightarrow x_\sigma = h^{-1} x_\mu g$

$$= \sum_\sigma x_\mu^{-1} \otimes h \cdot h^{-1} f(x_\mu g) = \sum_\sigma x_\mu^{-1} \otimes f(x_\mu g) = T(g(f)) \, . \qquad \blacksquare$$

So we have:

$$\mathrm{Ind}_H^G W \qquad\qquad \mathrm{Res}_H^G V$$

$$\mathrm{Hom}_{k[H]}(k[G]_1, W) \qquad\qquad \mathrm{Hom}_{k[G]}(k[G]_2, V) \text{ (exercise)}$$

$$k[G]_2 \underset{k[H]}{\otimes} W \qquad\qquad k[G]_1 \underset{k[G]}{\otimes} V$$

Reciprocity:
$$\mathrm{Hom}_{k[G]}\big(V,\ k[G]_2 \otimes_{k[H]} W\big) \simeq \mathrm{Hom}_{k[H]}\big(k[G]_1 \otimes_{k[G]} V,\ W\big)$$

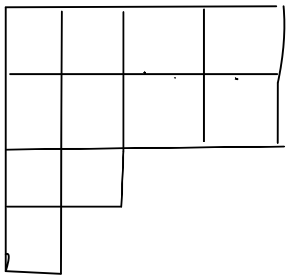
$$\mathrm{Hom}_{k[G]}\big(V,\ \mathrm{Hom}_{k[H]}(k[G], W)\big) \simeq \mathrm{Hom}_{k[H]}\big(\mathrm{Hom}_{k[G]}(k[G]_2, V), W\big)$$

Representations of the symmetric group S_n.

Def. $\lambda = (\lambda_1 \dots \lambda_r) \in (\mathbb{N}^*)^r$ is a partition of $n \in \mathbb{N}^*$ if $n = \lambda_1 + \lambda_2 + \dots + \lambda_r$,

where $\lambda_i \geq \lambda_{i+1} \geq 1$

Partitions of n $\longleftrightarrow$ Young diagrams of size n Y_λ



λ_1
λ_2
λ_3
λ_4

$\lambda = (4,4,2,1)$

$n = 11.$

λ_i = # of squares in the i-th row.

Def. A Young tableau T_λ is Y_λ filled with integers from 1 to n without repetitions.

1	6	7	5
2	9	3	8
4	11		
10			

T_λ

Def. Let (λ) be a partition of n, and choose a Young tableau T_λ

Let $P_\lambda \subset S_n$ the subgroup of permutations along the rows of T_λ.

$Q_\lambda \subset S_n$ the subgroup of permutations along the columns of T_λ.

Then $Q_\lambda \cap P_\lambda = 1$

Ex.

T_λ

1	2	3
4	5	
6	7	

$$P_\lambda = \underset{(1,2,3)}{S_3} \times \underset{(4,5)}{S_2} \times \underset{(6,7)}{S_2} = \left\{ \begin{matrix} 1 \\ (12)(13)(23) \\ (123)(132) \end{matrix}, \begin{matrix} 1 \\ (45) \end{matrix}, \begin{matrix} 1 \\ (67) \end{matrix} \right\}$$

$$|P_\lambda| = 6 \cdot 2 \cdot 2 = 24$$

$$Q_\lambda = \underset{(1,4,6)}{S_3} \times \underset{(2,5,7)}{S_3} = \left\{ \begin{matrix} 1 \\ (14)(16)(46) \\ (146)(164) \end{matrix}, \begin{matrix} 1 \\ (25)(27)(57) \\ (257)(275) \end{matrix} \right\}$$

$$|Q_\lambda| = 6 \cdot 6 = 36$$

Def. Young projectors.

Let $a_\lambda = \frac{1}{|P_\lambda|}\sum_{g\in P_\lambda} g$; $b_\lambda = \frac{1}{|Q_\lambda|}\sum_{g\in Q_\lambda}(-1)^g g$, where

$(-1)^g$ is the sign of $g\in S_n$

Let $x\in P_\lambda$

$$x a_\lambda = \frac{1}{|P_\lambda|}\sum_{g\in P_\lambda} xg = a_\lambda \Rightarrow a_\lambda^2 = \frac{1}{|P_\lambda|^2}\sum_{g\in P_\lambda} g\sum_{x\in P_\lambda} x = \frac{|P_\lambda|}{|P_\lambda|^2}\sum_{g\in P_\lambda} g = a_\lambda$$

$y\in Q_\lambda$:

$$y b_\lambda = \frac{1}{|Q_\lambda|}\sum_{g\in Q_\lambda}(-1)^g yg = \frac{1}{|Q_\lambda|}(-1)^y\sum_{g\in Q_\lambda}(-1)^{yg} yg = (-1)^y b_\lambda$$

$$\Rightarrow b_\lambda^2 = \frac{1}{|Q_\lambda|^2}\sum_{g\in Q_\lambda}(-1)^g g\sum_{y\in Q_\lambda}(-1)^y y = \frac{|Q_\lambda|}{|Q_\lambda|^2}\sum_{g\in Q_\lambda}(-1)^y(-1)^y gy = b_\lambda^2.$$

a_λ and b_λ are called the Young projectors.

Define $c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_n]$, $c_\lambda \neq 0$: coeff. of 1 is $\frac{1}{|P_\lambda||Q_\lambda|}$ in c_λ.

Def. A Specht module is defined as

$$V_\lambda = \mathbb{C}[S_n]c_\lambda \subset \mathbb{C}[S_n] \quad \text{is a left } \mathbb{C}[S_n]\text{-module}$$

Goal: to prove that (1) V_λ is irreducible

(2) Every irreducible representation of S_n is isomorphic to V_λ for a unique partition λ of n.

Remark: # of conjugacy classes of S_n = # of cycle types = # of partitions of n
= # of Young diagrams

Ex $\boxed{\qquad n \qquad}$ $Y_\lambda \Rightarrow a_\lambda = \frac{1}{|S_n|}\sum_{g\in S_n} g$; $Q_\lambda = \{1\} \Rightarrow b_\lambda = 1$

$P_\lambda = S_n$

$$\Rightarrow c_\lambda = a_\lambda b_\lambda = \frac{1}{|S_n|}\sum_{g\in S_n} g \quad ; \quad \mathbb{C}[S_n]c_\lambda = V_0 \text{ trivial representation}$$

$\begin{array}{|c|}\hline \\ n \\ \\ \hline\end{array}$ $\quad Y_n \Rightarrow P_\lambda = \{1\},\ a_\lambda = 1,\quad Q_\lambda = S_n \Rightarrow b_\lambda = \frac{1}{|S_n|}\sum_{g\in S_n} (-1)^g g$

$$x \in S_n \Rightarrow x\cdot c_\lambda = x\cdot b_\lambda = (-1)^x c_\lambda$$

$$\Rightarrow \mathbb{C}[S_n]\, c_\lambda = V_{sign} \quad \text{sign representation.}$$

S_3. $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad P_\lambda = \{1, (12)\},\ Q_\lambda = \{1, (13)\},\ a_\lambda = \frac{1}{2}(1+(12)),\ b_\lambda = \frac{1}{2}(1-(13))$

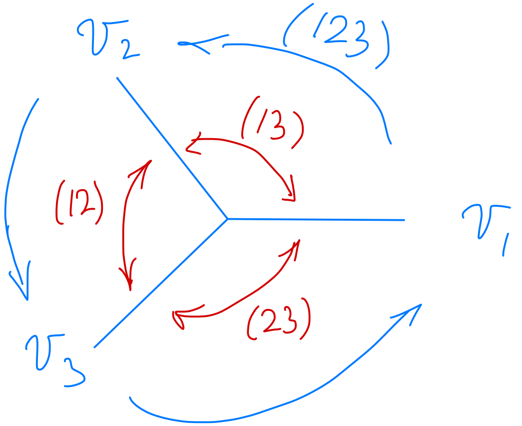
$$c_\lambda = \frac{1}{4}\big(1 + (12) - (13) - (12)(13)\big) = \frac{1}{4}\big(1 + (12) - (13) - (132)\big) = v_1$$

$$(123)\, c_\lambda = \frac{1}{4}\big((123) + (13) - (23) - 1\big) = v_2 \qquad (12)\, c_\lambda = c_\lambda = v_1$$

$$(132)\, c_\lambda = \frac{1}{4}\big((132) + (23) - (12) - (123)\big) = v_3 \qquad (13)\, c_\lambda = v_2$$

$$(23)\, c_\lambda = v_3$$

$$\Rightarrow v_1 + v_2 + v_3 = 0$$



$V_\lambda \simeq V_2$ the unique irreducible 2-dim representation of S_3.

Proof of the classification theorem.

Lemma 1. Let $x \in \mathbb{C}[S_n]$. Then $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, where $\ell_\lambda(x) \in \mathbb{C}$, and $\ell_\lambda(x+y) = \ell_\lambda(x) + \ell_\lambda(y)$.

Proof. (1) Let $g \in P_\lambda Q_\lambda$ meaning that $g = pq$, $p \in P_\lambda$, $q \in Q_\lambda$ uniquely.

Then $a_\lambda g b_\lambda = a_\lambda p q b_\lambda = (a_\lambda p)(q b_\lambda) = (-1)^q a_\lambda b_\lambda = (-1)^q c_\lambda$.

(2) Let $g \in S_n$, $g \neq pq$, $p \in P_\lambda$, $q \in Q_\lambda$. Then we will show that $a_\lambda g b_\lambda = 0$.

Suppose there exist a transposition $t = (ij)$ such that $t \in P_\lambda$ and $t \in g Q_\lambda g^{-1} \Leftrightarrow g t g^{-1} \in Q_\lambda$

Then $a_\lambda g b_\lambda = a_\lambda t g b_\lambda = a_\lambda g (g^{-1} t g) b_\lambda = a_\lambda g t' b_\lambda = -a_\lambda g b_\lambda = 0$.

$t' \in Q_\lambda$ a transposition

Let $T_\lambda' = g T_\lambda \Rightarrow Q_\lambda' = g Q_\lambda g^{-1}$

So we need to find i and j in the same row of T_λ and in the same column in $gT_\lambda = T_\lambda'$. We will show that if such i, j do not exist $\Rightarrow$ $g = pq \in P_\lambda Q_\lambda$.

If i, j do not exist $\Rightarrow$ any two elts in the 1st row of T_λ are in different columns in T_λ'.

$\Rightarrow \exists\, q_1' \in Q_\lambda'$ s.t. it moves all elts in the 1st row of T_λ to the 1st row of T_λ' and $\exists\, p_1 \in P_\lambda$ that reorders the elts in the 1st row of T_λ to match the elts in the 1st row of $q_1' T_\lambda'$.

$$\Rightarrow \quad (p_1 T_\lambda)_{1st\,row} = (q_1' T_\lambda')_{1st\,row}$$

If no i, j in the 2nd row of $p_1 T_\lambda$ are in the same column of $q_1' T_\lambda'$, then $\exists\, q_2' \in Q_\lambda'$ that moves all elts in the 2nd row of $p_1 T_\lambda$ to the 2nd row of $q_1' T_\lambda'$ and an elt $p_2 \in P_\lambda$ that reorders the elts of the 2nd row of $p_1 T_\lambda$ to match the 2nd row of $q_2' q_1' T_\lambda'$.

$$\Rightarrow (p_2 p_1 T)_{\text{1st and 2nd row}} = (q_2' q_1' T_\lambda')_{\text{1st and 2nd row}}$$

$\Rightarrow$ Continue until $\quad p T_\lambda = q' T_\lambda'$, $q' \in g Q_\lambda g^{-1}$, $T_\lambda' = g T_\lambda$

$$\Rightarrow \exists q \in Q_\lambda : \quad q' = g q g^{-1}.$$

$$\Rightarrow p T_\lambda = g q g^{-1} g T_\lambda \Rightarrow$$

$$T_\lambda = p^{-1} g q T_\lambda \Rightarrow p^{-1} g q = 1 \Rightarrow g = p q^{-1} \in P_\lambda Q_\lambda. \quad \blacksquare$$