Lecture 10

Recall: **Theorem.** Let G be a finite group and V an irreducible complex representation of G. Then $(\dim V)$ divides $|G|$.

Theorem (Burnside) A group of order $p^a q^b$, where p, q are primes and $a, b \geq 0$, is solvable.

Today: Induced representations. There is a method to construct a representation of G starting from a representation of a subgroup $H \subset G$.

If V is a complex representation of G, and $H \subset G$ is a subgroup, then V is also a representation of H by restriction:

$$\rho_V(H) = \rho_V(G)\Big|_H$$

If V is irreducible, the restriction $\mathrm{Res}_H^G V$ does not have to be irreducible.

Def. Let $H < G$ and V a representation of H. The *induced representation*

$$\mathrm{Ind}_H^G V = \{ f : G \to V : f(hx) = \rho_V(h) f(x) \ \forall x \in G, h \in H \}$$

with the action of G given by $g(f)(x) = f(xg)$ $\forall g \in G$.

This action is well defined:

$$g(g'(f))(x) = g'(f)(xg) = f(xgg') = (gg')(f)(x).$$

$$\mathrm{Ind}_H^G V \simeq \mathrm{Hom}_H(k[G], V) \quad \text{functions } f(h \cdot x) = \rho(h) f(x)$$

$$g(f)(x) = f(xg)$$

This action preserves $\mathrm{Ind}_H^G V$:

$$g(f)(hx) = f(hxg) = \rho(h) f(xg) = \rho(h) g(f)(x).$$

$$\dim \mathrm{Ind}_H^G V = \dim V \cdot \frac{|G|}{|H|}$$

Basis: $\{\delta_\sigma{}^i\}_{\sigma \in H\backslash G,\ i=1..\dim V}$

Let $\mu \in H\backslash G \Rightarrow \delta_\sigma{}^i(\mu) \overset{def}{=} \delta_{\sigma\mu} w_i$ where $\{w_i\}$ is a basis in V

Example. A_4 $\quad A_4 = \left\{ 1, \begin{matrix}(12)(34)\\(13)(24)\\(14)(32)\end{matrix}, \begin{matrix}(123)\\(214)\\(341)\\(432)\end{matrix}, \begin{matrix}(132)\\(241)\\(314)\\(423)\end{matrix} \right\}$

Let $K = \{1, (12)(34), (13)(24), (14)(23)\} \simeq \langle a, b : a^2 = b^2 = 1, ab = ba \rangle$

Right K-cosets: $\{ \underset{\delta_1}{K\cdot 1}, \underset{\delta_{123}}{K\cdot(123)}, \underset{\delta_{132}}{K\cdot(132)} \}$

Irreducible representations of K: V_0, $V_{\stackrel{+}{=}}$ $V_{\stackrel{\mp}{=}}$ $V_{\stackrel{=}{+}}$

$\mathrm{Ind}_K^{A_4} V_0 = \{\delta_1, \delta_{123}, \delta_{132}\}$ $\qquad \delta_{123}(214) = \delta_{123}((13)(24)(123)) =$

$= \rho(13)(24)\, \delta_{123}(123) = \rho(13)(24) \cdot 1$

$\rho_{\mathrm{Ind}_K^{A_4} V_0}((12)(34))\, \delta_1(x) = \delta_1(x(12)(34)) = \delta_1(x) \Rightarrow (12)(34) : \delta_1 \to \delta_1$

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$$— \quad '' \quad — \quad \delta_{123}(x) = \delta_{123}(x(12)(34)) = \delta_{123}(\underbrace{\cdots}_{\in K} x) = \underbrace{\rho(\underbrace{\cdots}_{\in K})}_{=1} \delta_{123}(x) \quad : \quad \delta_{123} \to \delta_{123}$$

$$\delta_{132} \qquad\qquad \delta_{132} \to \delta_{132}$$

$$\rho(12)(34) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\rho_{\mathrm{Ind}_K^{A_4} V_0}(123)\, \delta_1(x) = \delta_1(x(123)) = \delta_{132}(x) \quad \to \quad (123): \ \delta_1 \to \delta_{132}$$

$$\delta_{123}(x) = \delta_{123}(x(123)) = \delta_1(x) \qquad (123): \ \delta_{123} \to \delta_1$$

$$\delta_{132}(x) = \delta_{132}(x(123)) = \delta_{123}(x) \qquad (123): \ \delta_{132} \to \delta_{123}$$

$$\rho(123) = \begin{pmatrix} & 1 & 0 \\ & 0 & 1 \\ 1 & & \end{pmatrix}$$

Similarly $\rho(132)$ is a cyclic permutation of three basis elements

$$\Rightarrow \quad \chi_{\mathrm{Ind}_K^{A_4} V_0}((12)(34)) = 3 \qquad \chi_{\mathrm{Ind}_K^{A_4} V_0}(123) = \chi_{\mathrm{Ind}_K^{A_4} V_0}(132) = 0$$

A_4

	1	$(12)(34)$	(123)	(132)
$\mathrm{Ind}\, V_\pm$	3	-1	0	0
$\mathrm{Ind}\, V_0$	3	3	0	0
V_0	1	1	1	1
V_ζ	1	1	ζ	ζ^2
V_{ζ^2}	1	1	ζ^2	ζ
V_3	3	-1	0	0

$\zeta^3 = 1.$

$$\mathrm{Ind}_K^{A_4} V_\pm : \rho((12)(34))\,\delta_1(x) = \delta_1(x(12)(34)) = \delta_1(((12)(34))x) = \rho(\underbrace{(12)(34)}_{=1})\,\delta_1 = \delta_1 \qquad \delta_1 \to \delta_1$$

$$-\,''\,- \quad \delta_{123}(x) = \delta_{123}(x(12)(34)) = \delta_{123}((\overset{(14)(23)}{\cdots})x) = \rho((14)(23))\,\delta_{123} = -\delta_{123} \qquad \delta_{123} \to -\delta_{123}$$

$$x(12)(34) = (123) \Rightarrow x = (134) = (14)(23)(123)$$

$$-\,''\,- \quad \delta_{132}(x) \quad -\,''\,- \quad -\delta_{132} \qquad \delta_{132} \to -\delta_{132}$$

$$\Rightarrow \rho((12)(34)) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \Rightarrow \chi_{\rho_{\mathrm{Ind}_K^{A_4} V_\pm}}((12)(34)) = -1.$$

Action of $\rho(123)$, $\rho(132)$ same as for V_0.

Therefore, looking at the character table we can conclude that

$$\mathrm{Ind}_K^{A_4} V_0 \simeq V_0 \oplus V_{\zeta} \oplus V_{\zeta^2}.$$

$$\mathrm{Ind}_K^{A_4} V_{\pm} \simeq V_3$$

Frobenius formula for the character of an induced representation.

Theorem. Let $H \subset G$, let $\{x_\sigma\}_{\sigma \in H\backslash G}$ representatives of right H-cosets.

Then $$\chi(g) = \sum_{\sigma \in H\backslash G \,:\, x_\sigma g x_\sigma^{-1} \in H} \chi_V(x_\sigma g x_\sigma^{-1})$$

$\Updownarrow$

$x_\sigma g = h x_\sigma$ for $h \in H$

Proof: define $V_\sigma = \{ f \in \mathrm{Ind}_H^G V : f(g) = 0 \ \forall g \notin \sigma \}$

$\{\delta_\sigma^i\}_{\substack{\sigma \in H\backslash G \\ i=1\ldots \dim V}}$ basis in $V \Rightarrow \{\delta_\sigma^i\}_{i=1..\dim V}$ basis in V_σ

$$\Rightarrow \quad V = \bigoplus_{\sigma \in H\backslash G} V_\sigma, \quad \dim V_\sigma = \dim V$$

$\Rightarrow \chi(g) = \sum_\sigma \chi_\sigma(g)$ where $\chi_\sigma(g)$ is the trace of the diagonal block of $\rho(g)$ in V_σ.

Action $\rho_H^G(g) f(\sigma) = f(\sigma g)$ where σg is a right H-coset

$$tr_{V_\sigma}\, \rho_H^G(g) = 0 \quad \text{if} \quad \sigma \neq \sigma g$$

Assume $\sigma = \sigma g \Rightarrow x_\sigma g = h x_\sigma \Rightarrow h = x_\sigma g x_\sigma^{-1} \in H$

Define $\alpha : V_\sigma \to V$, $\alpha(f) = f(x_\sigma)$ isomorphism since f is determined by $f(x_\sigma)$

Consider the action of $G : V_\sigma \to V_\sigma$ and of $H : V \to V$

$$\rho_V(h) \cdot \alpha(f) = \rho_V(h) \cdot \underbrace{f(x_\sigma)}_{\in V} = f(h x_\sigma) = f(x_\sigma g) = \underbrace{\rho_H^G(g) f}(x_\sigma) = \alpha(\rho_H^G(g) f)$$

Let g: $x_\sigma g = h x_\sigma$

$$\Rightarrow \rho_H^G(g) f = \alpha^{-1} \circ \rho_V(h) \circ \alpha(f)$$

$$\Rightarrow \chi_\sigma(g) = tr_{V_\sigma}(\rho_H^G(g)) = tr_V(h) = \chi_V(h) \qquad \boxed{x_\sigma g x_\sigma^{-1} = h}$$

Sum up over the blocks $\chi(g) = \sum_{\sigma \in {}_H\backslash G,\ \sigma g = \sigma} \chi_V(x_\sigma g x_\sigma^{-1})$

$\blacksquare$

Another version: $\chi(g) = \frac{1}{|H|} \sum_{x \in G \,:\, xgx^{-1} \in H} \chi_V(xgx^{-1})$

<u>Example.</u> A_4, $K \subset A_4$, $\mathrm{Ind}_K^{A_4} V_0$, $\mathrm{Ind}_K^{A_4} V_\pm$ Characters:

$$\chi_{\mathrm{Ind} V}(g) = \sum_{\sigma \in K \backslash A_4 \,:\, \sigma g = \sigma} \chi_V(x_\sigma g x_\sigma^{-1})$$

$$\chi_{\mathrm{Ind} V}(1) = \chi_V(1) + \chi_V(1) + \chi_V(1) = 3\chi_V(1) = 3 \quad (3 \text{ cosets})$$

$$\chi_{\mathrm{Ind} V}((12)(34)) = \chi_V((12)(34)) + \chi_V((123)(12)(34)(132)) + \chi_V((132)(12)(34)(123)) =$$

$$= \chi_V((12)(34)) + \chi_V((14)(23)) + \chi_V((13)(24)) = \begin{cases} 3, & \mathrm{Ind}_K^{A_4} V_0 \\ -1, & \mathrm{Ind}_K^{A_4} V_\pm \end{cases}$$

$\chi_{\mathrm{Ind} V}((123))$: $\sigma \underset{(132)}{(123)} = \mu \neq \sigma \ \forall$ cosets

$$\Rightarrow \chi_{\mathrm{Ind} V_0, V_\pm}((123), (132)) = 0.$$

$\Rightarrow$ Get the same character as we computed before.

Frobenius reciprocity.

Theorem. $H \subset G$, V a representation of G, W a representation of H.

Then $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$

Remark: Restriction and induction are adjoint functors between $\mathcal{C}_{rep}(H)$ and $\mathcal{C}_{rep}(G)$.

Proof:

$$\begin{tikzcd} \mathrm{Hom}_G(V \arrow[r, bend left, "\alpha"] & \mathrm{Ind}_H^G W) \arrow[rr, bend right, "F"'] & & \mathrm{Hom}_H(\mathrm{Res}_H^G V \arrow[r, bend left, "\beta"] & W) \arrow[lll, bend left, "F'"] \end{tikzcd}$$

F: $F(\alpha)v \underset{\text{def}}{=} \alpha v(e)$ $e \in G$ identity

F': $(F'(\beta)v)(x) \underset{\text{def}}{=} \beta(xv)$ $x \in G$

$\boxed{1}$ $F(\alpha)$ is an H-intertwiner

$$F(\alpha)hv = \alpha h v(e) \underset{\alpha\in Hom_G}{=} (h\alpha v)(e) \underset{\alpha v\in Ind_H^G}{=} (\alpha v)(he) = (\alpha v)(eh) = h\cdot(\alpha v)(e) = hF(\alpha)v$$

$\boxed{2}$ $F'(\beta)v \in Ind_H^G W$

$$F'(\beta)v(hx) = \beta(hxv) \underset{\beta\in Hom_H}{=} h\beta(xv) = h\cdot(F'(\beta)v)(x)$$

$\boxed{3}$ $F'(\beta)$ is a G-intertwiner

$$F'(\beta)(gv)(x) = \beta(xgv) = F'(\beta)(xg) \underset{F'(\beta)\in Ind_H^G}{=} g\cdot F'(\beta)v(x)$$

$\boxed{4}$ $F\circ F' = Id_{Hom_H(\dots)}$

$$F(F'(\beta))v = (F'(\beta)v)(e) = \beta(ev) = \beta(v)$$

$\boxed{5}$ $F'' \circ F = Id_{Hom_G(\dots)}$

$$F'(F(\alpha)v)(x) = F(\alpha)(xv) = (\alpha xv)(e) = (x\alpha)v(e) = \alpha(v)(ex) = \alpha(v)(x)$$

$\alpha \in Hom_G$ $\qquad \alpha(v) \in Ind_G^H$ $\qquad \forall v \in V, x \in G.$

$\Rightarrow$ (1), (2) (3) show that we get a map between the correct spaces
(4), (5) show that they are bijective. ▨

Example. A_4 Reciprocity:

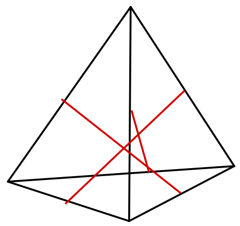
$$Hom_{A_4}(V_0, \underbrace{Ind_K^{A_4} V_0}_{\cong V_0 \oplus V_\zeta \oplus V_{\zeta^2}}) = Hom_K(\underbrace{Res_K^{A_4} V_0}_{\cong V_0}, V_0) \qquad dim = 1$$

$$Hom_{A_4}(V_\zeta, Ind_K^{A_4} V_0) = Hom_K(\underbrace{Res_K^{A_4} V_\zeta}_{\cong V_0}, V_0). \qquad dim = 1$$

$$Hom_{A_4}(V_\zeta, \underbrace{Ind_K^{A_4} V_\pm}_{\cong V_3}) = Hom_K(\underbrace{Res_K^{A_4} V_\zeta}_{\cong V_0}, V_\pm) = 0 \qquad dim = 0$$

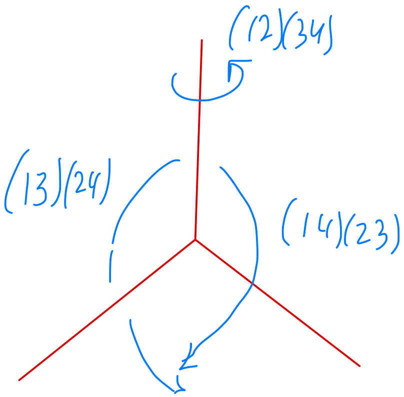
$$\mathrm{Hom}_{A_4}\left(\overline{V}_3,\ \underbrace{\mathrm{Ind}_K^{A_4} V_\pm}_{\simeq\, \overline{V}_3}\right) = \mathrm{Hom}_K\left(\underbrace{\mathrm{Res}_K^{A_4} \overline{V}_3}_{\simeq\, V_\pm \oplus V_\mp \oplus V_\mp},\ V_\pm\right) \qquad \text{dim } 1$$

$\overline{V}_3$:



A_4 acts by rotations

restrict to K



Vertical axis: $\rho(12)(34) = Id$

$\rho(13)(24) = -Id$

$\rho(14)(23) = -Id$

Similarly other 2 axes.

$$\Rightarrow \mathrm{Res}_K^{A_4} \overline{V}_3 \simeq V_\pm \oplus V_\mp \oplus V_\mp$$