

Throughout these solutions, $\bar{r}$ always denotes the class of an element r in some quotient.

Exercise 1.

We have to show that the $V(I)$'s satisfy axioms of a topology for the closed subsets (**1 pt**), namely

- (1) there exists $I \trianglelefteq R$ such that $V(I) = \emptyset$;
- (2) there exists $I \trianglelefteq R$ such that $V(I) = \text{Spec } R$;
- (3) for all ideals $I_1, \dots, I_n$, there exists $J \trianglelefteq R$ such that $V(J) = \cup_{j=1}^n V(I_j)$;
- (4) for all collections of ideals $\{I_j\}_{j \in A}$, there exists $J \trianglelefteq R$ such that $V(J) = \cap_{j \in A} V(I_j)$.

Let us show all this.

- (1) Since any prime ideal is by definition strictly contained in R , $V(R) = \emptyset$. (**1 pt**).
- (2) Since any ideal contains 0, $V((0)) = \text{Spec } R$. (**1 pt**).
- (3) Let $J = I_1 \cdots I_n$. Let us show that $V(J) = \cup_{j=1}^n V(I_j)$. (**1 pt**)
 Since $J \subseteq I_j$ for all j , we deduce that $V(I_j) \subseteq V(J)$ for all j , whence $\cup_{j=1}^n V(I_j) \subseteq V(J)$. (**1 pt**).
 On the other hand, if $p \in V(J)$ (i.e. $p \supseteq J$), assume by contradiction that for all j , $I_j \not\subseteq p$. Then for all j , there exists $a_j \in I_j$ such that $a_j \notin p$. However, $\prod_{j=1}^n a_j \in J$ by definition, so in particular $\prod_{j=1}^n a_j \in p$. Since p is prime, some $a_j \in p$, which is a contradiction. Thus, we also have $V(J) \subseteq \cup_{j=1}^n V(I_j)$. (**4 pts**).
- (4) Let $J = \sum_j I_j \trianglelefteq R$. We show that $V(J) = \cap_{j \in A} V(I_j)$. (**1 pt**)
 Since $I_j \subseteq J$ for all j , $V(J) \subseteq V(I_j)$ for all j , whence $V(J) \subseteq \cap_{j \in A} V(I_j)$. (**1 pt**).
 On the other hand, if $p \in \cap_{j \in A} V(I_j)$, then by definition $I_j \subseteq p$ for all j , so $J = \sum_{j \in A} I_j \subseteq p$, i.e. $p \in V(J)$. Thus, we also have $\cap_{j \in A} V(I_j) \subseteq V(J)$. (**1 pt**)

(Remove **1 pt** for those that write $V(I) \cap V(J) = V(I \cup J)$).

Exercise 2. (1) A *transcendence basis* of L over K is a subset $A \subseteq L$, such that A is algebraically independent over K and L is algebraic over $K(A)$. (**2 pts**)

- (2) Assume first that S is a field, and let $r \in R$ be a non-zero element. Since $S \hookrightarrow R$ is an integral extension, there exists $n \geq 0$ and $a_0, \dots, a_{n-1} \in S$ such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_0 = 0$$

(1 pt)

Assume n is minimal for this property. Then $a_0 \neq 0$, as otherwise

$$r(r^{n-1} + a_{n-1}r^{n-2} + \dots + a_1) = 0$$

and since $r \neq 0$ and R is a domain, also $n-1$ satisfies this property.

(1 pt)

Thus, $a_0 \neq 0$ so since S is a field, we obtain

$$\frac{-1}{a_0}(r^{n-1} + \dots + a_1)r = 1$$

Therefore, r is invertible. (2 pts)

Now, let us assume that R is a field, and let $s \in S$ be a non-zero element. Since R is a field, there exists $r \in R$ such that $rs = 1$, so we have to show that $r \in S$ (1 pt).

Since $S \hookrightarrow R$ is an integral extension, there exist $a_0, \dots, a_{n-1} \in S$ such that

$$r^n + a_{n-1}r^{n-1} + \dots + a_0 = 0$$

(1 pt).

Multiplying by s^{n-1} gives

$$r = -(a_{n-1} + a_{n-2}s + \dots + a_0s^{n-1})$$

Since the right-hand side is in S , we deduce that $r \in S$, so we win. (2 pts)

- (3) By Noether normalization, there exists a sub- F -algebra $S \subseteq R$ such that $S \cong F[t_1, \dots, t_r]$ as R -algebras and that $S \subseteq R$ is an integral extension. (2 pts)

Since $S \subseteq R$ induces an algebraic extension after passing to the fraction fields,

$$0 < \text{trdeg}_F(\text{Frac}(R)) = \text{trdeg}(\text{Frac}(S)) = \text{trdeg}_F(F(x_1, \dots, x_r)) = r$$

If R was a field, then so would be S by the previous point, and hence $F[x_1, \dots, x_r]$ would be a field. Since $r > 0$, this is impossible. (4 pts)

Exercise 3.

Let $(*)$ denote the things we are allowed to use that are stated in the exercise.

- (1) Let $p_0 \subseteq \dots \subseteq p_n = p$ be a strict chain of prime ideals of $T^{-1}R$. Then $p_0^c \subseteq \dots \subseteq p_n^c = p^c$ is also a chain of prime ideals (by $(*)$), and it is strict. Indeed, if $p_i^c = p_{i+1}^c$ for some $i < n$, then $p_i = p_i^{cc} = p_{i+1}^{cc} = p_{i+1}$ (the equalities follow from $(*)$) which contradicts the assumption. Therefore, by definition, $\text{ht}(p) \leq \text{ht}(p^c)$.

On the other hand, if $q_0 \subseteq \dots \subseteq q_n = p^c$ is a strict chain of prime ideals, then since $p^c \cap T = \emptyset$ (by $(*)$), also $q_i \cap T = \emptyset$ for all i . Hence, by $(*)$, we know that $q_0^e \subseteq \dots \subseteq q_n^e = p^{ce} = p$ is a chain of prime ideals. This chain is strict, because if $q_i^e = q_{i+1}^e$ for some $i < n$, then $q_i = q_i^{ec} = q_{i+1}^{ec} = q_{i+1}$ (again by $(*)$) so we have a contradiction. Thus, by definition $\text{ht}(p^c) \leq \text{ht}(p)$. **(4 pts)**

- (2) Recall that by definition, $R_p = T^{-1}R$ where $T = R \setminus p$. Since $p \cap (R \setminus p) = \emptyset$, we know by $(*)$ that p^e is a prime (and in particular strict) ideal. **(1 pt)**

Now, let us show that any element in $R_p \setminus p^e$ is invertible. Let $\frac{r}{s} \in R_p \setminus p^e$ with $r \in R$, $s \in S$. Since $\frac{r}{s} \notin R_p$, we know by $(*)$ that $r \notin p$. Therefore, $\frac{s}{r} \in R_p$, and by definition

$$\frac{s}{r} \frac{r}{s} = \frac{rs}{rs} = \frac{1}{1} = 1_{R_p}$$

Thus, $\frac{r}{s} \in R_p$ is invertible, so we have shown that $p^e \subseteq R_p$ is a strict ideal such that any element which is not in it is invertible. **(4 pts)**

Thus, any strict ideal has to be contained in p^e (otherwise it would contain an invertible element), so p^e is necessarily the unique maximal ideal of R_p , i.e. R_p is local with unique maximal ideal p^e . **(2 pts)**

- (3) Since R_p is local with maximal ideal, it is immediate from the definitions that $\dim(R_p) = \text{ht}(p^e)$. On the other hand, by the first point we see that $\text{ht}(p^e) = \text{ht}(p^{ec})$ so we conclude since $p^{ec} = p$ by $(*)$. **(3 pts)**

Exercise 4.

Let $\gamma_{B'} \in B'$. Since c is surjective, there exists $\gamma_C \in C$ such that $c(\gamma_C) = f'_2(\gamma_{B'})$. But then,

$$d(f_3(\gamma_C)) = f'_3(c(\gamma_C)) = f'_3(f'_2(\gamma_{B'})) = 0$$

so since d is injective, $f_3(\gamma_C) = 0$. **(4 pts)**

By exactness, there exists $\gamma_B \in B$ such that $f_2(\gamma_B) = \gamma_C$, so we obtain

$$f'_2(\gamma_{B'} - b(\gamma_B)) = f'_2(\gamma_{B'}) - f'_2(b(\gamma_B)) = f'_2(\gamma_{B'}) - c(f_2(\gamma_B)) = f'_2(\gamma_{B'}) - c(\gamma_C) = 0$$

By exactness, there exists $\gamma_{A'} \in A'$ such that $f'_1(\gamma_{A'}) = \gamma_{B'} - b(\gamma_B)$. **(4 pts)**

By surjectivity of a , there exists γ_A such that $a(\gamma_A) = \gamma_{A'}$. Therefore, we finally obtain

$$\begin{aligned} b(f_1(\gamma_A) + \gamma_B) &= b(f_1(\gamma_A)) + b(\gamma_B) \\ &= f'_1(a(\gamma_A)) + b(\gamma_B) \\ &= f'_1(\gamma_{A'}) + b(\gamma_B) \\ &= \gamma_{B'} - b(\gamma_B) + b(\gamma_B) \\ &= 0 \end{aligned}$$

so $\gamma_{B'} \in \text{im}(b)$. (4 pts)

Exercise 5. (1) Let us show that $\dim_{\mathbb{R}}(M) = 4$, by showing that $\{\bar{1}, \bar{x}, \bar{y}, \bar{xy}\}$ forms a basis (2 pts). First note that $(x^2, x^2 + y^2) = (x^2, y^2)$

Linearly independent : If not, there would exist $(a, b, c, d) \neq (0, 0, 0, 0)$ in $\mathbb{R}^4$ such that

$$a + bx + cy + dxy = \gamma x^2 + \delta y^2$$

for some $\gamma, \delta \in R[x, y]$. Since the combination is non-trivial, the right-hand side is non-zero. Therefore it has to have degree ≥ 2 in either x or y , but this is not the case for the left-hand side. Thus we get a contradiction. (2 pts)

They generate M : Given $p(x, y) \in \mathbb{R}[x, y]$, we can write it as $p(x, y) = a + bx + cy + dxy + q_1x^2 + q_2y^2$ for $q_1, q_2 \in R[x, y]$ and real numbers a, b, c, d . Therefore, in the quotient by (x^2, y^2) , $\bar{p}$ is an $\mathbb{R}$ -linear combination of $\{\bar{1}, \bar{x}, \bar{y}, \bar{xy}\}$ so we win. (1 pt)

Let us now show that $\dim_{\mathbb{R}}(N) = 4$. As $\mathbb{R}$ -vector spaces (in fact even as $\mathbb{R}$ -algebras),

$$\mathbb{R}[x, y]/(x^2, y^2 + 1) \cong (\mathbb{R}[y]/(y^2 + 1)) / (x^2) \cong \mathbb{C}[x]/(x^2)$$

This $\mathbb{C}$ -vector space has dimension 2 (same argument as what we did for M), so the underlying $\mathbb{R}$ -vector space has dimension 4. (5 pts).

If the student proceeds again to show linear independence and generation with an explicit basis, 1 pt for the basis and 2 pts for both generation and linear independence. In particular, if the student just gave both basis, they get 3 pts.

- (2) We have seen in the solutions of the first hand-in exercise (exercise 6 in sheet 2) that if k is any field, the length of $k[x, y]/(y^n, y^m)$ as a $k[x, y]$ -module is nm . Thus, here we get that the length is 4.

One could also show it directly : since the dimension is 4, the length cannot be greater than 4 (an $\mathbb{R}[x, y]$ -module is in particular an $\mathbb{R}$ -vector space). So all we have to do is to exhibit a chain of submodules

of length 4. Consider

$$(0) \subseteq (\overline{xy}) \subseteq (\overline{x}) \subseteq (\overline{x}, \overline{y}) \subseteq M$$

We just have to show that all inclusions are strict.

Since $\overline{xy}$ is part of a basis, it cannot be 0, so the first inclusion is strict.

For the second one, if it was not strict, then we would obtain

$$x = \gamma xy + \beta_1 x^2 + \beta_2 y^2$$

for some $\beta_1, \beta_2, \gamma \in \mathbb{R}[x, y]$. Setting $y = 0$ and comparing degrees shows that this is impossible.

For the third one, if this was not strict, we would obtain

$$y = \gamma x + \beta_1 x^2 + \beta_2 y^2$$

for some $\gamma, \beta_1, \beta_2 \in \mathbb{R}[x, y]$. Setting $x = 0$ and comparing degrees gives again a contradiction.

For the fourth one, if it was not strict, then we could write

$$1 = \alpha x + \beta y + \gamma x^2 + \delta y^2$$

for some $\alpha, \beta, \delta, \gamma \in \mathbb{R}[x, y]$. Setting $x = y = 0$ gives the sought contradiction. **(5 pts : 2 pts for giving the chain and 3 points for the explanation)**

- (3) Since $y^2 + 1$ acts as 0 on N , N is naturally a $\mathbb{R}[x, y]/(y^2 + 1)$ -module, and being a $\mathbb{R}[x, y]$ -submodule of N is exactly the same as being a $\mathbb{R}[x, y]/(y^2 + 1)$ -submodule. Now there is a ring isomorphism

$$\mathbb{R}[x, y]/(y^2 + 1) \cong \mathbb{C}[x]$$

sending x to x and y to i , and under this isomorphism of rings, N corresponds to the $\mathbb{C}[x]$ -module $\mathbb{C}[x]/(x^2)$ (this is because x^2 is sent to x^2 in the above ring isomorphism). Thus, the length of N as an $\mathbb{R}[x, y]$ -module is exactly the same as the length of $\mathbb{C}[x]/(x^2)$ as a $\mathbb{C}[x]$ -module. We have seen in exercise 4.1 of sheet 2 that the length of this module is 2 (one could also say that $(0) \subseteq (\overline{x}) \subseteq \mathbb{C}[x]/(x^2)$ is a composition series, by the same argument as what we did for M). **(5 pts)**

If the student gives a chain and shows it works, 1 point for giving the chain and 4 for showing it works

Exercise 6. (1) Let M be a simple module. As we have seen in exercise 1.2 in sheet 1, $M \cong S/\text{Ann}(m)$ for some $m \in M$. **(1 pt)**.

Since $\text{Ann}(m)$ is an ideal, it has to be either maximal or the whole ring S (otherwise this would contradict the simplicity of M by the

correspondence theorem). Thus we have shown that any simple module is of the form S/I where I is either maximal or S .

Since $\text{Ann}_S(S/I) = I$, we deduce that

$$M \cong S/\text{Ann}_S(M)$$

In particular we immediately conclude that if M and N are two simple modules with $\text{Ann}_S(M) = \text{Ann}_S(N)$, then $M \cong N$. **(2 pts)**

To see the converse, note that in general, if $M \cong N$ for any modules M, N , then $\text{Ann}_S(M) = \text{Ann}_S(N)$. Indeed, let $\theta : M \rightarrow N$ be an isomorphism. Then if $a \in \text{Ann}_S(N)$, we have for all $m \in M$, $0 = a\theta(m) = \theta(am)$ so by injectivity of θ , $am = 0$. Thus, $a \in \text{Ann}_S(M)$. Doing the same with θ^{-1} proves the claim. **(1 pt)**

- (2) Since $M \neq 0$, we know by the above discussion that $M \cong R/I$ for I a maximal ideal of R . We conclude by the weak Nullstellensatz (by assumption, F is algebraically closed). **(2 pts)**
- (3) Note that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of finite length R -modules, then by exercise 2 of sheet 2,

$$\text{length}(B) = \text{length}(A) + \text{length}(C)$$

and by basic linear algebra,

$$\dim_F(B) = \dim_F(A) + \dim_F(C)$$

(we do not know yet if the dimensions are finite, so we set $\infty = \infty + \infty$). Thus, by induction on the length, all we have to do is to show that if M is a non-zero simple module, $\dim_F(M) = 1$. **(3 pts)**

By the previous point, such an M is isomorphic (as an R -module, so in particular as an F -vector space) to

$$R/(x_1 - c_1, \dots, x_n - c_n)$$

for some $c_1, \dots, c_n \in F$. Let us show that $\{1\}$ is a basis. It is linearly independent since $1 \notin (x_1 - c_1, \dots, x_n - c_n)$ (evaluate at $x_1 = c_1, \dots, x_n = c_n$ to see this), so we are left to show that it generates $R/(x_1 - c_1, \dots, x_n - c_n)$. Given $p(x_1, \dots, x_n)$, write

$$p(x_1 + c_1, \dots, x_n + c_n) = a + \alpha_1 x_1 + \dots + \alpha_n x_n$$

with $\alpha_i \in R$, $a \in F$. Then

$$p(x_1, \dots, x_n) = a + \sum_i \alpha_i (x_1 - c_1, \dots, x_n - c_n)(x_i - c_i)$$

so in the quotient, is it a multiple of 1. **(1 pt)**

- (4) Let $R := F[x, y]$ and $M = F[x, y]/(x, y)$. We have seen in Example 4.1.2 of the notes that a projective resolution of M is

$$0 \rightarrow R \xrightarrow{\alpha} R^{\oplus 2} \xrightarrow{\beta} R$$

where $\alpha(r) = (yr, -xr)$ and $\beta(r, s) = rx + ys$. (**1 pt**)

Let us compute $\text{Hom}_R(\alpha, M)$ and $\text{Hom}_R(\beta, M)$. In fact, we claim that both maps are 0.

Let us see it for $\text{Hom}_R(\alpha, M)$. Given $\gamma : R^{\oplus 2} \rightarrow M$, we have

$$\gamma(\alpha(r)) = \gamma(yr, -xr) = y\gamma(r, 0) - x\gamma(0, r)$$

However, both x and y act as 0 on M by definition, so

$$\text{Hom}_R(\alpha, M)(\gamma) = \gamma \circ \alpha = 0$$

(**3 pts**)

In fact, the same argument shows that $\text{Hom}(\beta, M) = 0$, so

$$\text{Ext}_R^1(M, M) \cong \text{Hom}_R(R^{\oplus 2}, M) \cong M^{\oplus 2}$$

Let us show the last isomorphism. We know that $\text{Hom}_R(-, M)$ is additive, so it is enough to show that $\text{Hom}_R(R, M) \cong M$. The following maps are inverses : $\text{Hom}_R(R, M) \rightarrow M$ sending f to $f(1)$ and the map $M \rightarrow \text{Hom}_R(R, M)$ sending m to $f : R \rightarrow M$ defined by $f(r) = rm$.

(**1 pt**)

- (5) Let $a \in F$, and consider the extension

$$0 \rightarrow M \xrightarrow{\delta} M_a \xrightarrow{\mu} M \rightarrow 0$$

where $M_a = R/(x^2, xy, x - ya)$, $\delta : M \rightarrow M_a$ is defined by

$$\delta(\bar{r}) = \overline{xr}$$

(this is well-defined since $\bar{x} \in M_a$ is (x, y) -torsion by definition) and $\mu : M_a \rightarrow M$ is the canonical map coming from the inclusion of ideals $(x^2, xy, y - xa) \subseteq (x, y)$ (i.e. the factorization of the quotient map $R \rightarrow R/(x, y)$ through M_a). (**3 pts**)

Let us first show this is exact. Since $\mu : M_a \rightarrow M$ is the factorization of a surjective map, it is also surjective. Furthermore, $\mu \circ \delta = 0$ since $\mu(\bar{x}) = 0$, so we have $\ker(\mu) \supseteq \text{im}(\delta)$. Let us show the other inclusion. Assume $\mu(\bar{r}) = 0$, then by definition we can write $r = \lambda_1 x + \lambda_2 y$ ($r \in R$ denotes a lift of $\bar{r}$). In M_a , $\bar{y} = -\bar{x}a$, so $\bar{r} = \delta(\overline{\lambda_1 - \lambda_2 a})$. Thus, $\ker(\mu) = \text{im}(\delta)$. Finally, let us show that δ is not the zero map. This will immediately conclude its injectivity since M is simple.

If δ was the zero map, then by the above μ would be an isomorphism, or in other words $(x^2, xy, y - xa) = (x, y)$. In particular we would have $x \in (x^2, xy, y - xa)$, so we could write $x = \lambda_1 x^2 + \lambda_2 xy + \lambda_3(y - xa)$. Since x and $y - xa$ are coprime, we must have that x divides λ_3 (write $\lambda_3 = x\lambda'_3$). Then dividing by x gives

$$1 = \lambda_1 x + \lambda_2 y + \lambda'_3(y - xa)$$

Putting $x = y = 0$ in the above gives $1 = 0$, which is a contradiction.

Thus we have finally proven that these sequences are exact. **(3 pts)**

Now we are left to show that these extensions are not trivial, that $M_a \not\cong M_b$ for all $a \neq b \in F$ and that F is infinite. The latter statement holds since F is algebraically closed (a finite field k cannot be algebraically closed, because the polynomial $\prod_{a \in k}(t - a) + 1$ has no root). **(1 pt)**

To see the first statement, if this extension was trivial, then we would automatically have $M_a \cong M^{\oplus 2}$. But then, by what we did in the first point, their annihilators would agree. This would give $(x^2, xy, y - xa) = (x, y)$. We have already proven this does not hold before. **(2 pts)**

To see that for all $a \neq b \in F$, $M_a \not\cong M_b$, it is again enough to show that their annihilators are different, i.e. that

$$(x^2, xy, y - xa) \neq (x^2, xy, y - xb)$$

Let us show that $y - xa \notin (x^2, xy, y - xb)$. By contradiction, since $y - xa = y - xb + (a - b)x$, we would obtain that $(a - b)x \in (x^2, xy, y - xb)$. But $a - b \in F \setminus \{0\} \subseteq R^\times$, so we would further obtain $x \in (x^2, xy, y - xb)$. We have already proven this is not the case (when showing that δ was injective) so we win. **(2 pts)**

If the student uses the explicit description of the isomorphism between the first Ext-group and the Yoneda extensions, then they get 4 points, and 7 are dedicated to showing the rest : 1 for the infinite field, 6 points to explain why the modules are pairwise non-isomorphic.