Rings and modules (MATH-311) — Final exam — Solutions

29 January 2022, 8 h 15 – 11 h 15

The exam consisted of five exercises. The first four were worth 16 points each and the last one was worth 36 points. So there was a total of 100 points in the exam. It was possible to solve any point of an exercise assuming the statements of the previous points, even if not all of those were solved. Also, a partial solution to a point gives partial credit.

Exercise 1 [**16 pts**]

In the next exercise you can use without proof the following statement: let $\gamma : A \to B$ and $\xi : A \to C$ be two R-module homomorphisms and assume that γ is surjective. Then:

$$\xi \in \operatorname{im} \operatorname{Hom}_R(\gamma, C) \iff \ker \xi \supseteq \ker \gamma$$

(1) Show that for an arbitrary ring R and an R-module N, the functor $\operatorname{Hom}_R(\_, N)$ is left exact, which means (by definition) that for every short exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} K \longrightarrow 0$$

the following sequence is exact

$$\operatorname{Hom}_R(L, N) \xleftarrow{\operatorname{Hom}_R(\alpha,N)} \operatorname{Hom}_R(M, N) \xleftarrow{\operatorname{Hom}_R(\beta,N)} \operatorname{Hom}_R(K, N) \longleftarrow 0$$

(2) Give an example where $\operatorname{Hom}_R(\alpha, N)$ in the above sequence is not surjective.

Solution.

(1) [10 pts] We call SES the short exact sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} K \to 0$. We want to show that $\operatorname{Hom}_R(\text{SES}, N)$ is exact at $\operatorname{Hom}_R(K, N)$ and $\operatorname{Hom}_R(M, N)$. Note that $\operatorname{Hom}_R(\alpha, N)$ resp. $\operatorname{Hom}_R(\beta, N)$ are just precomposition by α resp. β.

Exactness at $\operatorname{Hom}_R(K, N)$*:* [5 pts] We want to show that the kernel of $\operatorname{Hom}_R(\beta, N)$ is equal to the image of $0 \to \operatorname{Hom}_R(K, N)$, i.e. that $\operatorname{Hom}_R(\beta, N)$ is injective. Let $\phi \in \ker \operatorname{Hom}_R(\beta, N)$ be arbitrary, i.e. $\phi \circ \beta = 0$. Then $\operatorname{im} \beta \subseteq \ker \phi$, but by exactness of SES at K we have that $\operatorname{im} \beta = K$. Thus $\phi = 0$, and we conclude $\ker \operatorname{Hom}_R(\beta, N) = 0$. This proves exactness at $\operatorname{Hom}_R(K, N)$.

Exactness at $\operatorname{Hom}_R(M, N)$*:* [5 pts] Notice that as $\operatorname{Hom}_R(\_, N)$ is a contravariant functor we have

$$\operatorname{Hom}_R(\alpha, N) \circ \operatorname{Hom}_R(\beta, N) = \operatorname{Hom}_R(\underbrace{\beta \circ \alpha}_{=0}, N) = 0$$

where for the last equality we used that precomposition by the zero-map is the zero-map. Hence $\operatorname{im} \operatorname{Hom}_R(\beta, N) \subseteq \ker \operatorname{Hom}_R(\alpha, N)$ [2 pts], so it remains to show the reverse inclusion. Let $\psi \in \ker \operatorname{Hom}_R(\alpha, N)$ be arbitrary, i.e. $\psi \circ \alpha = 0$. By exactness of SES at M, we obtain

$$\ker \beta \overset{\text{SES}}{=} \operatorname{im} \alpha \subseteq \ker \psi.$$

As $\beta : M \to K$ is surjective, the statement in the beginning of the exercise implies that $\psi \in \operatorname{im} \operatorname{Hom}_R(\beta, N)$. As $\psi \in \ker \operatorname{Hom}_R(\alpha, N)$ was arbitrary, we conclude $\operatorname{im} \operatorname{Hom}_R(\beta, N) \supseteq \ker \operatorname{Hom}_R(\alpha, N)$ [3 pts], and thus $\operatorname{im} \operatorname{Hom}_R(\beta, N) = \ker \operatorname{Hom}_R(\alpha, N)$.

(2) [6 pts]: 3 pts for giving a right counterexample and 3 pts for its justification We take $R = L = M = N = \mathbb{Z}$, $K = \mathbb{Z}/2\mathbb{Z}$, $\alpha : L \to M$ multiplication by 2, and $\beta : M \to K$ the natural projection map. Then $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} K \to 0$ is clearly exact. Now consider $\mathrm{id}_{\mathbb{Z}} \in \operatorname{Hom}_R(L, N)$ and suppose by contradiction that there exists $\psi \in \operatorname{Hom}_R(M, N)$ with $\mathrm{id}_{\mathbb{Z}} = \operatorname{Hom}_R(\alpha, N)(\psi)$, i.e. $\mathrm{id}_{\mathbb{Z}} = \psi \circ \alpha$. Then

$$1 = \mathrm{id}_{\mathbb{Z}}(1) = \psi(\alpha(1)) = \psi(2) = 2\psi(1).$$

This is a contradiction as 1 is odd. So $\operatorname{Hom}_R(\alpha, N)$ isn't surjective.

Exercise 2 [**16 pts**]

Let F be a field.

In this exercise you can use without proof the following:

- *for $c_i \in F$, the ideal $m_{c_1,\ldots,c_n} = (x_1 - c_1, \ldots, x_n - c_n) \subseteq F[x_1, \ldots, x_n]$ is a maximal ideal.*
- *If R is a finitely generated commutative F-algebra, then $\dim R = \operatorname{trdeg}_F \operatorname{Frac}(R)$.*

(1) Show that if F is algebraically closed and $m \subseteq F[x_1, \ldots, x_n]$ is a maximal ideal, then $m = m_{c_1,\ldots,c_n}$ for some $c_i \in F$.

(2) Give a counterexample to the previous point if F is not algebraically closed.

Solution.

(1) [10 pts] Let $R = {F[x_1, \ldots, x_n]}\big/{m}$, then R is a finitely generated commutative F-algebra, as it is a quotient of a polynomial ring over F in finitely many variables. Note that as m is maximal, R is a field, and that the structure of R as an F-algebra is given by the inclusion $\iota : F \to R$, defined by $\iota(c) := c + m \in R$ for all $c \in F$ (it is injective because field morphisms are injective). As R is a field, the only prime ideal is (0), and thus

$$0 = \dim R = \operatorname{trdeg}_F R$$

where we used $\operatorname{Frac}(R) = R$ ([5 pts] for proving that R/F is a field extension with $\operatorname{trdeg}_F R = 0$). Hence $F \overset{\iota}{\hookrightarrow} R$ is an algebraic extension. So as F is algebraically closed, it must be the trivial extension, i.e. ι is an isomorphism [2 pts]. For $1 \leq i \leq n$, let $c_i := \iota^{-1}(x_i + m) \in F$. Then by definition

$$c_i + m = \iota(c_i) = x_i + m \implies x_i - c_i \in m$$

for all $1 \leq i \leq n$, and thus $m_{c_1,\ldots,c_n} \subseteq m$. But then as $m_{c_1,\ldots,c_n}$ is maximal and m is a proper ideal, we obtain $m_{c_1,\ldots,c_n} = m$ [3 pts].

(2) [6 pts]: 3 pts for giving a right counterexample and 3 pts for its justification Take $F = \mathbb{R}$ and $m = (x^2+1) \subseteq \mathbb{R}[x]$. Notice that $x^2+1 \in \mathbb{R}[x]$ is irreducible, because it is a quadratic polynomial without roots (and if it were a product of smaller degree polynomials, i.e. of degree 1, it would have to have real roots). Thus as $\mathbb{R}[x]$ is a PID, m is a maximal ideal. But two ideals (f) and (g) in $\mathbb{R}[x]$ are equal if and only if f and g differ by a unit, and so in particular f and g must have the same degree. Hence m isn't of the form m_c for any $c \in \mathbb{R}$.

Exercise 3 $\left[\ \textbf{16 pts}\ \right]$

(1) Let R be an arbitrary ring. Show that given a short exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

such that M' and M'' have finite length over R, the following equality holds:

$$\operatorname{length}_R M = \operatorname{length}_R M' + \operatorname{length}_R M''$$

(2) If F is a field, and $f \in F[x]$ is a polynomial that is a product of $n > 0$ irreducible polynomials, then $\operatorname{length}_{F[x]} \left(F[x] \big/ (f) \right) = n$.

Solution.

(1) [8 pts] Let $l' = \operatorname{length}_R M'$, $l'' = \operatorname{length}_R M''$ and $l := l' + l''$. Also, let $0 = M_0' \subseteq \cdots \subseteq M_{l'}' = M'$ and $0 = M_0'' \subseteq \cdots \subseteq M_{l''}'' = M''$ be composition series. By replacing M' with its image inside M we may assume that $M' \subseteq M$, and then we may also replace M'' by M / M'. Let $\pi : M \twoheadrightarrow M''$ be the natural projection map. Now for $0 \le i \le l$, define

$$M_i := \begin{cases} M_i' & \text{if } i \le l' \\ \pi^{-1}(M_i'') & \text{if } i \ge l'. \end{cases}$$

This is well defined because $M_{l'}' = M' = \pi^{-1}(0) = \pi^{-1}(M_{l'}'')$. It is then also clear that $M_0 = 0$, $M_l = M$ and $M_i \subseteq M_{i+1}$ for all $i < l$ ([4 pts] For constructing a composition series for M out of compositions series of M', M'', including valid justifications on how the contruction works.). Now if $i < l'$ then $M_{i+1} / M_i = M_{i+1}' \big/ M_i'$ is simple, and if $l' \le i < l$, then

$$M_{i+1} / M_i \cong M_{i+1}/M' \big/ M_i / M' = \pi(M_{i+1}) \big/ \pi(M_i) = M_{i+1}'' \big/ M_i''$$

is simple as well, where we used the 3rd isomorphism theorem and surjectivity of π. Hence $0 = M_0 \subseteq \cdots \subseteq M_l = M$ is a composition series for M ([4 pts] For justifying that the proposed composition series is indeed a composition series, including the use of the 3rd isomorphism theorem.), and thus

$$\operatorname{length}_R M = l = l' + l'' = \operatorname{length}_R M' + \operatorname{length}_R M''.$$

(2) [8 pts] We present two different solutions.

Solution 1. We proceed by induction on n. If $n = 1$, then f is irreducible, and thus $(f) \subseteq F[x]$ is maximal (as $F[x]$ is a PID). As the $F[x]$-submodules of $F[x] \big/ (f)$ are in one-to-one correspondence with the $F[x]$-submodules of $F[x]$ containing (f), which by maximality are precisely (f) and $F[x]$, we obtain that $F[x] \big/ (f)$ is simple, and hence $\operatorname{length}_{F[x]} \left(F[x] \big/ (f) \right) = 1$ [2 pts]. Now assume the statement to be true for a fixed $n \in \mathbb{Z}_{>0}$, and let f be a product of $n+1$ irreducible polynomials. Let p be an irreducible factor of f and write $f = pg$ where g is a product of n irreducible polynomials. Now as the projection $F[x] \to F[x] \big/ (g)$ has f in its kernel, we obtain an induced (surjective) map ${F[x]}\big/{(f)} \to {F[x]}\big/{(g)}$, defined by $h+(f) \mapsto h+(g)$. Its kernel is precisely ${(g)}\big/{(f)}$, which is the image of the map $F[x] \to {F[x]}\big/{(f)}$ defined by $h \mapsto gh+(f)$. Notice that the kernel of this latter map is precisely (p), because for all $h \in F[x]$ we have

$$gh \in (f) \iff f = gp \mid gh \iff p \mid h \iff h \in (p).$$

Hence we obtain an induced injective map ${F[x]}\big/{(p)} \to {F[x]}\big/{(f)}$ with image precisely the kernel of ${F[x]}\big/{(f)} \to {F[x]}\big/{(g)}$. Hence we have a short exact sequence of $F[x]$-modules

$$0 \longrightarrow {F[x]}\big/{(p)} \longrightarrow {F[x]}\big/{(f)} \longrightarrow {F[x]}\big/{(g)} \longrightarrow 0,$$

([4 pts] for the construction of that exact sequence) and thus by point (1) and the induction hypothesis it follows that

$$\operatorname{length}_{F[x]}\left({F[x]}\big/{(f)}\right) = \operatorname{length}_{F[x]}\left({F[x]}\big/{(p)}\right) + \operatorname{length}_{F[x]}\left({F[x]}\big/{(g)}\right) = n+1.$$

[2 pts]

Solution 2. [8 pts]: 4pts fot the construction of a normal series and 4 pts for its justification Let $p_1, \dots, p_n \in F[x]$ be irreducible such that $f = p_1 \cdots p_n$. For $0 \le i \le n$ define $M_i := \left(\prod_{i \le j \le n} p_j\right)\Big/{(f)}$ (where the empty product is taken to be equal to 1 by convention). Hence in particular $M_0 = {(f)}\big/{(f)} = 0$ and $M_n = {(1)}\big/{(f)} = {F[x]}\big/{(f)}$. Also, it is clear that $M_i \subseteq M_{i+1}$ for all $i < n$. Furthermore, we have by the 3rd isomorphism theorem

$$M_{i+1}/M_i \cong \left(\prod_{i+1 \le j \le n} p_j\right)\Big/\left(\prod_{i \le j \le n} p_j\right)$$

for all $0 \le i < n$. Now the $F[x]$-submodules of $\left(\prod_{i+1 \le j \le n} p_j\right)\Big/\left(\prod_{i \le j \le n} p_j\right)$ are in one-to-one correspondence with the $F[x]$-submodules of $\left(\prod_{i+1 \le j \le n} p_j\right)$ containing $\left(\prod_{i \le j \le n} p_j\right)$. That is, the ideals $(g) \subseteq F[x]$ contained in $\left(\prod_{i+1 \le j \le n} p_j\right)$, containing $\left(\prod_{i \le j \le n} p_j\right)$. That is, up to multiplication by a unit, the polynomials $g \in F[x]$ divisible by $\prod_{i+1 \le j \le n} p_j$, dividing $\prod_{i \le j \le n} p_j$. As a PID is a UFD, the polynomials satisfying this are precisely $\prod_{i+1 \le j \le n} p_j$ and $\prod_{i \le j \le n} p_j$, up to multiplication by a unit. In conclusion, the only submodules of $\left(\prod_{i+1 \le j \le n} p_j\right)\Big/\left(\prod_{i \le j \le n} p_j\right)$ are the zero module and the module itself, and thus M_{i+1}/M_i is simple. Hence $0 = M_0 \subseteq \cdots \subseteq M_n = {F[x]}\big/{(f)}$ is a composition series of ${F[x]}\big/{(f)}$, and thus we conclude $\operatorname{length}_{F[x]}\left({F[x]}\big/{(f)}\right) = n$.

Exercise 4 [**16 pts**]

We have learned during the course that for a an arbitrary ring R and for a short exact sequence of co-chain complexes of R-modules

$$0 \longrightarrow F_\bullet \xrightarrow{\alpha_\bullet} G_\bullet \xrightarrow{\beta_\bullet} H_\bullet \longrightarrow 0$$

there are R-module homomorphism $\delta_i : H^i(H_\bullet) \to H^{i+1}(F_\bullet)$ for every integer i, such that the following long exact sequence is exact:

$$\dots \xrightarrow{\delta_{i-1}} H^i(F_\bullet) \xrightarrow{H^i(\alpha_\bullet)} H^i(G_\bullet) \xrightarrow{H^i(\beta_\bullet)} H^i(H_\bullet) \xrightarrow{\delta_i} H^{i+1}(F_\bullet) \xrightarrow{H^{i+1}(\alpha_\bullet)} \dots$$

Here, a short exact sequence of co-chain complexes means that $\alpha_\bullet$ and $\beta_\bullet$ are co-chain morphisms such that

$$0 \longrightarrow F_i \xrightarrow{\alpha_i} G_i \xrightarrow{\beta_i} H_i \longrightarrow 0$$

is exact for every $i \in \mathbb{Z}$.

(1) Define δ_i.

(2) Show that δ_i are well defined R-module homomorphisms.

Solution.

We denote by $f_i : F_i \to F_{i+1}$, $g_i : G_i \to G_{i+1}$ resp. $h_i : H_i \to H_{i+1}$ the connecting morphisms. The short exact sequence $0 \to F_i \xrightarrow{\alpha_i} G_i \xrightarrow{\beta_i} H_i \to 0$ will be called SES_i.

(1) [6 pts]: 4pts for chasing $z \mapsto x$ and 2pts for showing $x \in \ker f_{i+1}$ δ_i has to be a map from $H^i(H_\bullet) = \ker h_i / \operatorname{im} h_{i-1}$ to $H^{i+1}(F_\bullet) = \ker f_{i+1} / \operatorname{im} f_i$. So let $z + \operatorname{im} h_{i-1} \in H^i(H_\bullet)$ be arbitrary. By exactness of SES_i, β_i is surjective, and thus there exists $y \in G_i$ such that $\beta_i(y) = z$. But then as $z \in \ker h_i$ we have

$$\beta_{i+1}(g_i(y)) = h_i(\beta_i(y)) = h_i(z) = 0.$$

Thus $g_i(y) \in \ker \beta_{i+1} = \operatorname{im} \alpha_{i+1}$, where we used exactness of SES_{i+1}. Thus there exists $x \in F_{i+1}$ with $\alpha_{i+1}(x) = g_i(y)$. Notice that we then have

$$\alpha_{i+2}(f_{i+1}(x)) = g_{i+1}(\alpha_{i+1}(x)) = g_{i+1}(g_i(y)) = 0.$$

By exactness of SES_{i+2} this implies $f_{i+1}(x) = 0$, and thus $x \in \ker f_{i+1}$. We then pose $\delta_i(z + \operatorname{im} h_{i-1}) := x + \operatorname{im} f_i \in H^{i+1}(F_\bullet)$.

(2) [10 pts] To prove that δ_i is well-defined, we have to prove that the target element constructed in the previous point doesn't depend on any of the choices involved [4 pts]. To do so, we follow the construction step by step. So let $z, z' \in \ker h_i$ be such that $z + \operatorname{im} h_{i-1} = z' + \operatorname{im} h_{i-1}$. Then there exists $c \in H_{i-1}$ such that

$$z = z' + h_{i-1}(c).$$

Now as in the previous point, let $y, y' \in G_i$ be such that $\beta_i(y) = z$ and $\beta_i(y') = z'$. Also, by exactness of SES_{i-1}, let $b \in G_{i-1}$ be such that $\beta_{i-1}(b) = c$. We then have

$$\beta_i(y) = \beta_i(y') + h_{i-1}(\beta_{i-1}(b)) = \beta_i(y') + \beta_i(g_{i-1}(b)).$$

So by exactness of SES_i, there exists $a \in F_i$ such that

$$y = y' + g_{i-1}(b) + \alpha_i(a).$$

Finally, as in the previous point, let $x, x' \in F_{i+1}$ be such that $\alpha_{i+1}(x) = g_i(y)$ and $\alpha_{i+1}(x') = g_i(y)$. By applying g_i to the above equation, we obtain

$$\alpha_{i+1}(x) = g_i(y) = g_i(y') + \underbrace{g_i(g_{i-1}(b))}_{=0} + g_i(\alpha_i(a)) = \alpha_{i+1}(x') + \alpha_{i+1}(f_i(a)).$$

By exactness of SES_{i+1}, α_{i+1} is injective, and thus $x = x' + f_i(a)$. In particular, we obtain $x + \operatorname{im} f_i = x' + \operatorname{im} f_i$. Thus δ_i is a well-defined set-map.

So it remains to prove that δ_i is R-linear. To this end, let $z + \operatorname{im} h_{i-1},\ z' + \operatorname{im} h_{i-1} \in H^i(H_\bullet)$ and $r, r' \in R$ be arbitrary, and let $z'' := rz + r'z'$. We again follow the construction in (1) step by step: first, let $y, y' \in G_i$ be such that $\beta_i(y) = z$ and $\beta_i(y') = z'$, then $y'' := ry + r'y' \in G_i$ is such that $\beta_i(y'') = z''$. Now as in (1), there exist $x, x' \in \ker f_i$ such that $\alpha_{i+1}(x) = g_i(y)$ and $\alpha_{i+1}(x') = y'$. So $x'' := rx + r'x' \in \ker f_i$ is such that $\alpha_{i+1}(x'') = g_i(y'')$. Hence, as δ_i is well-defined, we have

$$\begin{aligned}\delta_i(r(z + \operatorname{im} h_{i-1}) + r'(z' + \operatorname{im} h_{i-1})) &= \delta_i(z'' + \operatorname{im} h_{i-1}) = x'' + \operatorname{im} f_i = \\ &= r(x + \operatorname{im} f_i) + r'(x' + \operatorname{im} f_i) = r\delta_i(z + \operatorname{im} h_{i-1}) + r'\delta_i(z' + \operatorname{im} h_{i-1}).\end{aligned}$$

Thus δ_i is a morphism of R-modules [4 pts].

Exercise 5 $\left[\text{ 36 pts }\right]$

Let R be a commutative ring.

In this exercise, you can use the following statements we proved throughout the course: if R is Artinian, then

- *every prime ideal of R is maximal, and*
- $\operatorname{length}_R R < \infty$.

This exercise is about the interplay between the notions Noetherian and Artinian for rings. To help distinguishing between them, we set with bald font the appearances of the word Artinian. Also, recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

(1) Show that if R is Noetherian, $m \subseteq R$ a maximal ideal, and $I \subseteq R$ is an m-primary ideal, then there is an integer $j > 0$ such that $m^j \subseteq I$. Give a counterexample to this statement when R is not Noetherian.

(2) Show that if $j > 0$ is an integer, $m \subseteq R$ is a maximal ideal and R is Noetherian, then R/m^j is an **Artinian** local ring.

(3) Show that if R is Noetherian of dimension 0, then there exists and integer $j > 0$ and maximal ideals $m_1, \dots, m_s$ of R such that

$$(0) = \bigcap_{i=1}^{s} m_i^j.$$

Deduce the ring isomorphism $R \cong \prod_{i=1}^{s} R_i$, where R_i are **Artinian** local rings.

(4) Show that if R is **Artinian** if and only if it is Noetherian and $\dim R = 0$.

(5) Show that if R is **Artinian**, and $T \subseteq R$ is a multiplicatively closed set, then $T^{-1}R$ is also **Artinian**.

Solution.

(1) [8 pts] As R is Noetherian, there exist $r_1, \dots, r_l \in R$ such that $m = (r_1, \dots, r_l)$ [2 pts]. Now as $\sqrt{I} = m$, let $N \in \mathbb{Z}_{>0}$ be large enough such that $r_k^N \in I$ for all $1 \le k \le l$, and let $j := Nl + 1$ [2 pts]. Let $x_1, \dots, x_j \in m$ be arbitrary, then there exist $a_{ik} \in R$ for $1 \le i \le j$ and $1 \le k \le l$ such that $x_i = \sum_k a_{ik} r_k$ for all i. Then by expanding

$$x_1 \cdots x_j = \left(\sum_k a_{1k} r_k \right) \cdots \left(\sum_k a_{jk} r_k \right),$$

every summand has at least one r_i which appears with an exponent greater than or equal to N, and hence every summand is in I. Thus $x_1 \cdots x_j \in I$ [2 pts] and

$$m^j = (x_1 \cdots x_j \mid x_1, \dots, x_j \in m) \subseteq I.$$

As a counterexample for the case where R is not Noetherian, take $R = F[x_i \mid i \in \mathbb{Z}_{>0}] = F[x_1, x_2, \dots]$ where F is a field, $I = (x_1^2, x_2^2, x_3^2, \dots)$ and $m = (x_1, x_2, \dots)$. Notice that m is maximal because $R/m \cong F$ is a field. Also, $x_i \in \sqrt{I}$ for all i, and thus as $\sqrt{I}$ is a proper ideal we have $m = \sqrt{I}$, i.e. I is m-primary (as m is maximal). Finally, suppose by contradiction that $m^j \subseteq I$ for some $j > 0$. Then in particular $x_1 \cdots x_j \in I$, and thus there exists elements $a_1, \ldots, a_l \in R$ such that

$$x_1 \cdots x_j = a_1 x_1^2 + \cdots + a_l x_l^2.$$

This is a contradiction, as no term of the form $a_k x_k^2$ can contain the monomial $x_1 \cdots x_j$ with non-zero coefficient [2 pts].

(2) [8 pts] We will make heavy use of the following elementary fact that was used several times either explicitly or implicitly throughout the course:

Fact. Let R be a commutative ring and $I \subseteq R$ and ideal.

- If M is an R-module such that $I \subseteq \mathrm{Ann}_R(M)$, then M naturally has the structure of an ${}^{R}\!/_{I}$-module (via the formula $(r + I)m := rm$), and the ${}^{R}\!/_{I}$-submodules of M are precisely the R-submodules.
- If M is an ${}^{R}\!/_{I}$-module, then it has naturally the structure of an R-module (via the formula $rm := (r + I)m$) with $I \subseteq \mathrm{Ann}_R(M)$, and the R-submodules of M are precisely the ${}^{R}\!/_{I}$-submodules.
- The constructions of the two previous points are mutually inverse.

The proofs are almost tautological. As a consequence of the above, if M is an R-module with $I \subseteq \mathrm{Ann}_R(M)$, or an ${}^{R}\!/_{I}$-module, then $\mathrm{length}_R(M) = \mathrm{length}_{R/I}(M)$, simply because the submodules stay the same. For the same reason, ${}^{R}\!/_{I}$ is Artinian/Noetherian as an R-module if and only if it is Artinian/Noetherian as a ring.

Now to the exercise; we proceed by induction on j. If $j = 1$, then as $m \subseteq R$ is maximal, ${}^{R}\!/_{m}$ is a field, and thus in particular an Artinian local ring [2 pts]. Now assume the statement to be true for a fixed $j \in \mathbb{Z}_{>0}$. Notice that the natural surjection $R \twoheadrightarrow {}^{R}\!/_{m^j}$ has m^{j+1} in its kernel, and thus induces a surjection ${}^{R}\!/_{m^{j+1}} \twoheadrightarrow {}^{R}\!/_{m^j}$ defined by $r + m^{j+1} \mapsto r + m^j$. Thus the kernel is precisely ${}^{m^j}\!\big/_{m^{j+1}}$, and we obtain a short exact sequence of R-modules [2 pts]

$$0 \to {}^{m^j}\!\big/_{m^{j+1}} \to {}^{R}\!/_{m^{j+1}} \to {}^{R}\!/_{m^j} \to 0.$$

Now as Noetherianity is stable under taking submodules and quotients, ${}^{m^j}\!\big/_{m^{j+1}}$ is a Noetherian R-module. Also, notice that $m \subseteq \mathrm{Ann}_R\left({}^{m^j}\!\big/_{m^{j+1}}\right)$, and thus it naturally has the structure of a Noetherian $k := {}^{R}\!/_{m}$-module. As k is a field, we thus have that ${}^{m^j}\!\big/_{m^{j+1}}$ is a Noetherian k-vector space, i.e. a finite dimensional k-vector space. But thus it is also an Artinian k-vector space, and so it is also Artinian as an R-module. Consequently, as both ${}^{m^j}\!\big/_{m^{j+1}}$ and ${}^{R}\!/_{m^j}$ are Artinian R-modules, ${}^{R}\!/_{m^{j+1}}$ is an Artinian R-module as well, and hence also an Artinian ring [2 pts].

It remains to see that ${}^{R}\!/_{m^{j+1}}$ is local [2 pts]. The prime ideals of ${}^{R}\!/_{m^{j+1}}$ are in one-to-one correspondence with the prime ideals of R containing m^{j+1}. But a prime ideal containing m^{j+1} must contain m (if it doesn't contain $x \in m$ then it neither contains $x^{j+1} \in m^{j+1}$ by the primality condition). Hence the only prime ideal of ${}^{R}\!/_{m^{j+1}}$ is ${}^{m}\!/_{m^{j+1}}$, and thus it is a local ring.

(3) [8 pts] As R is Noetherian, primary decompositions exist. So let $(0) = \bigcap_{i=1}^{s} I_i$ be a primary decomposition of the trivial ideal. As R is of dimension 0, every prime ideal of

R is maximal, and thus $m_i := \sqrt{I_i}$ is maximal for $1 \leq i \leq s$. By point (1), there exists $j > 0$ such that $m_i^j \subseteq I_i$ for all $1 \leq i \leq s$, and thus

$$(0) = \bigcap_{i=1}^{s} m_i^j \subseteq \bigcap_{i=1}^{s} I_i = (0),$$

hence $\bigcap_{i=1}^{s} m_i^j = (0)$ [4 pts].

To conclude this part of the exercise, we would like to use the Chinese remainder theorem. To this end, we have to prove that $m_1^j, \ldots, m_s^j$ are pairwise coprime. Let $i \neq i'$, then by maximality there exist $x \in m_i$ and $x' \in m_{i'}$ such that $x + x' = 1$. Thus we also have

$$1 = (x + x')^{2j} = \sum_{0 \leq l \leq 2j} \binom{2j}{l} x^l (x')^{2j-l}.$$

So if $y := \sum_{j \leq l \leq 2j} \binom{2j}{l} x^l (x')^{2j-l}$ and $y' := \sum_{0 \leq l < j} \binom{2j}{l} x^l (x')^{2j-l}$ then $y \in m_i$ and $y' \in m_{i'}$ are such that $y + y' = 1$. Hence $m_1^j, \ldots, m_s^j$ are pairwise coprime, and thus by the CRT we have

$$R \cong R/\textstyle\bigcap_{i=1}^{s} m_i^j \overset{\text{CRT}}{\cong} R/m_i^j.$$

Finally, by point (2), $R_i := R/m_i^j$ is an Artinian local ring for all $1 \leq i \leq s$, so we are done [4 pts].

(4) [8 pts] If R is Artinian, then by the statement in the beginning of the exercise we have $\text{length}_R R < \infty$, so R is also Noetherian. Furthermore, as every prime ideal of R is maximal, we have $\dim R = 0$, so we already proved the 'only if' implication [4 pts].

To prove the reverse implication [4 pts], let R be a Noetherian ring of dimension 0. By point (3), we have a ring isomorphism $R \cong \prod_{i=1}^{s} R_i$ such that $R_1, \ldots R_s$ are Artinian local rings. So to conclude, it remains to see that a product of Artinian rings is Artinian. Notice that if $I \subseteq \prod_{i=1}^{s} R_i$ is an ideal and $I_{i'} \subseteq R_{i'}$ its contraction under the inclusion $R_{i'} \to \prod_{i=1}^{s} R_i$, then $I = \prod_{i=1}^{s} I_i$. Indeed, denote by $e_{i'} \in \prod_{i=1}^{s} R_i$ the vector whose components are all zero except for the i'-th component which is equal to 1. Then if $(x_1, \ldots, x_s) \in I$, we have $(0, \ldots, 0, x_{i'}, 0, \ldots, 0) = e_{i'} \cdot (x_1, \ldots, x_s) \in I$, and thus $x_{i'} \in I_{i'}$, which proves $(x_1, \ldots, x_s) \in \prod_{i=1}^{s} I_i$. On the other hand, if $(y_1, \ldots, y_s) \in \prod_{i=1}^{s} I_i$ is arbitrary, then as $y_{i'} \in I_{i'}$ it follows that $e_{i'}(y_1, \ldots, y_s) \in I$. Hence

$$(y_1, \ldots, y_s) = \sum_{i=1}^{s} e_i (y_1, \ldots, y_s) \in I.$$

So we indeed have $I = \prod_{i=1}^{s} I_i$.

Now let $(I_l)_{l \in \mathbb{Z}_{>0}}$ be a decreasing sequence of ideals of $\prod_{i=1}^{s} R_i$. Write $I_l = \prod_{i=1}^{s} I_{li}$ for all $l > 0$, then for all $1 \leq i \leq s$ we have that $(I_{li})_{l \in \mathbb{Z}_{>0}}$ is a decreasing sequence of ideals of R_i. As all the R_i's are Artinian, we can choose $L > 0$ large enough such that $I_{li} = I_{Li}$ for all $l \geq L$ and $1 \leq i \leq s$. Hence $I_l = I_L$ for all $l \geq L$, i.e. $(I_l)_{l \in \mathbb{Z}_{>0}}$ stabilizes. So we conclude that $R \cong \prod_{i=1}^{s} R_i$ is Artinian.

(5) [4 pts] Let $(J_l)_{l \in \mathbb{Z}_{>0}}$ be a decreasing sequence of ideals of $T^{-1}R$. Then $(J_l^c)_{l \in \mathbb{Z}_{>0}}$ is a decreasing sequence of ideals of R and thus it stabilizes, i.e. there exists $L > 0$ such that $J_l^c = J_L^c$ for all $l \geq L$. Hence $J_l = J_l^{ce} = J_L^{ce} = J_L$ for all $l \geq L$, i.e. $(J_l)_{l \in \mathbb{Z}_{>0}}$ stabilizes. Hence $T^{-1}R$ is Artinian.