Rings and modules (MATH-311) — Final exam

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Name : Grothendieck Alexander

SCIPER : 42

Signature : ______________________________

Numéro

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Paper & pen: This booklet contains 120 exercises, on 28 pages, for a total of 2400 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of the additional paper. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper handwritten by yourself. At the end, we collect the cheat sheets.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Results of the course: You can use all results seen during the lectures or in the exercise sessions (that is, all results in the lectures notes or on the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, please state explicitly what you are using, and why the assumptions are satisfied.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: all rings are with identity.

Question:	1	2	3	4	5	Total
Points:	16	16	16	16	36	100
Score:						

Exercise 1 [**16 pts**]

In this exercise you can use without proof the following statement: let R be a ring, and let $\gamma : A \to B$ and $\xi : A \to C$ be two R-module homomorphisms such that γ is surjective. Then:

$$\xi \in \operatorname{im} \operatorname{Hom}_R(\gamma, C) \iff \ker \xi \supseteq \ker \gamma$$

(1) Show that for an arbitrary ring R and an R-module N, the functor $\operatorname{Hom}_R(\_, N)$ is left exact, which means (by definition) that for every short exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} K \longrightarrow 0$$

the following sequence is exact

$$\operatorname{Hom}_R(L, N) \xleftarrow{\operatorname{Hom}_R(\alpha,N)} \operatorname{Hom}_R(M, N) \xleftarrow{\operatorname{Hom}_R(\beta,N)} \operatorname{Hom}_R(K, N) \longleftarrow 0$$

(2) Give an example where $\operatorname{Hom}_R(\alpha, N)$ in the above sequence is not surjective.

Exercise 2 [**16 pts**]

Let F be a field.

In this exercise you can use without proof the following:

- *For $c_i \in F$, the ideal $m_{c_1,\dots,c_n} = (x_1 - c_1, \dots, x_n - c_n) \subseteq F[x_1, \dots, x_n]$ is a maximal ideal.*
- *If R is a finitely generated commutative F-algebra, then $\dim R = \operatorname{trdeg}_F \operatorname{Frac}(R)$.*

(1) Show that if F is algebraically closed and $m \subseteq F[x_1, \dots, x_n]$ is a maximal ideal, then $m = m_{c_1,\dots,c_n}$ for some $c_i \in F$.

(2) Give a counterexample to the previous point if F is not algebraically closed.

Exercise 3 [**16 pts**]

(1) Let R be an arbitrary ring. Show that given a short exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

such that M' and M'' have finite length over R, the following equality holds:

$$\operatorname{length}_R M = \operatorname{length}_R M' + \operatorname{length}_R M''$$

(2) If F is a field, and $f \in F[x]$ is a polynomial that is a product of $n > 0$ irreducible polynomials, then $\operatorname{length}_{F[x]} \left(F[x] \big/ (f) \right) = n$.

Exercise 4 [**16 pts**]

We have learned during the course that for a an arbitrary ring R and for a short exact sequence of co-chain complexes of R-modules

$$0 \longrightarrow F_\bullet \xrightarrow{\alpha_\bullet} G_\bullet \xrightarrow{\beta_\bullet} H_\bullet \longrightarrow 0$$

there are R-module homomorphism $\delta_i : H^i(H_\bullet) \to H^{i+1}(F_\bullet)$ for every integer i, such that the following long exact sequence is exact:

$$\dots \xrightarrow{\delta_{i-1}} H^i(F_\bullet) \xrightarrow{H^i(\alpha_\bullet)} H^i(G_\bullet) \xrightarrow{H^i(\beta_\bullet)} H^i(H_\bullet) \xrightarrow{\delta_i} H^{i+1}(F_\bullet) \xrightarrow{H^{i+1}(\alpha_\bullet)} \dots$$

Here, a short exact sequence of co-chain complexes means that $\alpha_\bullet$ and $\beta_\bullet$ are co-chain morphisms such that

$$0 \longrightarrow F_i \xrightarrow{\alpha_i} G_i \xrightarrow{\beta_i} H_i \longrightarrow 0$$

is exact for every $i \in \mathbb{Z}$.

(1) Define δ_i.

(2) Show that δ_i is a well defined R-module homomorphisms.

Exercise 5 [**36 pts**]

Let R be a commutative ring.

In this exercise, you can use the following statements we proved throughout the course: if R is Artinian, then

- *every prime ideal of R is maximal, and*
- $\operatorname{length}_R R < \infty$.

This exercise is about the interplay between the notions Noetherian and Artinian for rings. To help distinguishing between them, we set with bald font the appearances of the word Artinian. Also, recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

(1) Show that if R is Noetherian, $m \subseteq R$ is a maximal ideal, and $I \subseteq R$ is an m-primary ideal, then there is an integer $j > 0$ such that $m^j \subseteq I$. Give a counterexample to this statement when R is not Noetherian.

(2) Show that if $j > 0$ is an integer, $m \subseteq R$ is a maximal ideal and R is Noetherian, then R/m^j is an **Artinian** local ring.

(3) Show that if R is Noetherian of dimension 0, then there exists and integer $j > 0$ and maximal ideals $m_1, \ldots, m_s$ of R such that

$$(0) = \bigcap_{i=1}^{s} m_i^j.$$

Deduce the ring isomorphism $R \cong \prod_{i=1}^{s} R_i$, where R_i are **Artinian** local rings.

Hint: you can use without proof that the Chinese remainder theorem holds for any number of ideals, as soon as the considered ideals are pairwise coprime.

(4) Show that R is **Artinian** if and only if it is Noetherian and $\dim R = 0$.

(5) Show that if R is **Artinian**, and $T \subseteq R$ is a multiplicatively closed set, then $T^{-1}R$ is also **Artinian**.