General information for the Rings and modules (MATH-311) final
(18.01.2021, 16:15-19:15)

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1. exercise

Does $R = \mathbb{C}[x,y] \big/ (x^5, y^7)$ have finite length (as a module over itself)? If not, then prove it. If yes, then compute its length.

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2. exercise

(a) Let p and q be two prime ideals of a ring R. Show that $S = R \setminus (p \cup q)$ is a multiplicatively closed set of R.

Take $R = \mathbb{Z}$ from now, and $S = \mathbb{Z} \setminus \big((2) \cup (3)\big)$.

(b) Show that $S^{-1}R$ is a PID.

(c) Find all the prime ideals of $S^{-1}R$.

(d) Find all the primary ideals of $S^{-1}R$.

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3. exercise

For a prime $p > 2$, set $R = \mathbb{F}_p[x] \big/ (x^p)$. Compute for every integer $i \geq 0$ the R-module

$$\operatorname{Ext}^i_R \left(\mathbb{F}_p[x] \big/ (x), \mathbb{F}_p[x] \big/ (x^2) \right),$$

where $\mathbb{F}_p[x] \big/ (x^i)$ is endowed with an R-module structure via the natural surjection $\mathbb{F}_p[x] \big/ (x^p) \twoheadrightarrow \mathbb{F}_p[x] \big/ (x^i)$ for $i = 1, 2$. Explain carefully all the steps of your computation.

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4. exercise

(a) For a field k, we define $R = k[\![x]\!]$ as we defined $\mathbb{F}_q[\![x]\!]$ in Exercise 4 of Sheet 9. That is, $k[\![x]\!]$ is the set of formal power series $\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in k$, and addition and multiplication goes as it goes for polynomials. With formulas the operations are:

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) + \left(\sum_{i=0}^{\infty} b_i x^i \right) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i \right) = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} a_i b_{j-i} \right) x^i$$

You do not have to show the fact that R with the above operations is a commutative ring with unity.

Show then that R is a PID. Describe the units of R and the prime elements of R.

(b) Find a direct sum M of free cyclic R-modules and of cyclic R-modules with prime power annihilators such that M is isomorphic as an R-module to the quotient module:

$$R \oplus R \oplus R \Big/ R \cdot \big(1 + x, 1, x\big) + R \cdot \big(x, x^3, x^4\big)$$

$$= R \oplus R \oplus R \Big/ \left\{ r \cdot \big(1 + x, 1, x\big) + s \cdot \big(x, x^3, x^4\big) \;\middle|\; r, s \in R \right\}$$

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5. exercise

(a) Show that if $0 \neq h \in \mathbb{C}[x, y]$ is a prime element, then $\mathbb{C}[x,y]\big/(h)$ is not a field.

(b) Show that if $f \in R$ is a (non-unit and non-zero) prime element of a Noetherian domain, then the only prime ideal properly contained in (f) is (0) (in particular the height of (f) is 1).

(c) Show that if $p \subseteq R$ is a height 1 prime ideal in a Noetherian UFD, then $p = (g)$, where $g \in R$ is a prime element.

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