Solutions for the exam Rings and modules (MATH-311) final
(18.01.2021, 16:15-19:15)

The exam consist of 5 exercises, each exercise is worth 20 points and the total worth of the exam is 100 points.

**Exercise 1** (20 points)**.**
Does $R = \mathbb{C}[x,y]\Big/(x^5, y^7)$ have finite length (as a module over itself)? If not, then prove it. If yes, then compute its length.

*Proof.* First we note that since $R$ is a 35-dimensional vector-space over $\mathbb{C}$ (a basis is given by $x^i y^j$ for $0 \leq i \leq 4$ and $0 \leq j \leq 6$) it is of finite length equal to 35 as a $\mathbb{C}$-vectorspace. We claim that it is easy to see that this implies that $R$ is of finite length as an $R$-module. To this end let

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = R$$

be a strictly increasing chain of ideals in $R$. By definition, this defines a strictly increasing sequence of $\mathbb{C}$-vectorspaces. By exercise sheet 2, exercise 4 (3) this implies that $n \leq 35$.

A *correct* proof that $R$ has finite length has been awarded with 10 points

We now construct a composition series of $R$ as follow:

$$0 = (\bar{x}^5\bar{y}^7) \subsetneq (\bar{x}^4\bar{y}^6) \subsetneq (\bar{x}^4\bar{y}^5) \subsetneq \cdots \subsetneq (\bar{x}^4\bar{y})$$
$$(\bar{x}^4\bar{y}) \subsetneq (\bar{x}^4\bar{y}, \bar{x}^3\bar{y}^6) \subsetneq (\bar{x}^4\bar{y}, \bar{x}^3\bar{y}^5) \subsetneq \cdots \subsetneq (\bar{x}^4\bar{y}, \bar{x}^3\bar{y}) = (\bar{x}^3\bar{y})$$
$$(\bar{x}^3\bar{y}) \subsetneq (\bar{x}^3\bar{y}, \bar{x}^2\bar{y}^6) \subsetneq (\bar{x}^3\bar{y}, \bar{x}^2\bar{y}^5) \subsetneq \cdots \subsetneq (\bar{x}^3\bar{y}, \bar{x}^2\bar{y}) = (\bar{x}^2\bar{y})$$
$$(\bar{x}^2\bar{y}) \subsetneq (\bar{x}^2\bar{y}, \bar{x}\bar{y}^6) \subsetneq (\bar{x}^2\bar{y}, \bar{x}\bar{y}^5) \subsetneq \cdots \subsetneq (\bar{x}^2\bar{y}, \bar{x}\bar{y}) = (\bar{x}\bar{y})$$
$$(\bar{x}\bar{y}) \subsetneq (\bar{x}\bar{y}, \bar{y}^6) \subsetneq (\bar{x}\bar{y}, \bar{y}^5) \subsetneq \cdots \subsetneq (\bar{x}\bar{y}, \bar{y}) = (\bar{y})$$
$$(\bar{y}) \subsetneq (\bar{y}, \bar{x}^4) \subsetneq (\bar{y}, \bar{x}^3) \subsetneq \cdots \subsetneq (\bar{y}, \bar{x}) \subsetneq R$$

It is evident that $(\bar{x}, \bar{y}) \in \mathrm{Ann}(M_i/M_{i-1})$ for all i (in fact the way to come up with the above series is to make sure that $\bar{x}M_i \subset M_{i-1}$ and $\bar{y}M_i \subset M_{i-1}$ in each step), in particular $\mathrm{Ann}(M_i/M_{i+1})$ is a maximal ideal and hence the quotients $M_i/M_{i+1}$ are simple for all $i$. The above is a composition series with all quotients isomorphic to $R/(\bar{x}, \bar{y}) \cong \mathbb{C}$. Note that also the fact that the chain has length 35 directly implies that it has to be a composition series since if the quotients where not simple the above chain could be extended to a chain with strictly more terms, a contradiction to what was proven in the previous paragraph.

Providing a correct composition series gives 20 points (since in particular this implies that $R$ has finite length).

□

**Grading Scheme 1.**
The following has been applied when awarding points to incomplete or incorrect solutions:

- A *correct* proof that $R$ has finite length has been awarded with 10 points
- A "proof" that $R$ has finite length by means of providing an incorrect composition series has been awarded 5 points

**Exercise 2** (20 points)**.** 1. Let $p$ and $q$ be two prime ideals of a ring $R$. Show that $S = R \setminus (p \cup q)$ is a multiplicatively closed set of $R$. 5 points

Take $R = \mathbb{Z}$ from now, and $S = \mathbb{Z} \setminus \big((2) \cup (3)\big)$.

2. Show that $S^{-1}R$ is a PID. 5 points
3. Find all the prime ideals of $S^{-1}R$. 5 points
4. Find all the primary ideals of $S^{-1}R$. 5 points

*Proof.* 1. We check that

- $1 \in S$, since $1 \notin p$ and $1 \notin q$. 1 point
- if $s, r \in S$ then $sr \in S$. This is true since $s \notin P$ and $r \notin P$ implies $sr \notin P$ for any primeideal $P$. 4 points

2. We know that $R = \mathbb{Z}$ is a PID. By Prop. 6.3.9 (1) in the course notes every ideal of $S^{-1}R$ is extended i.e., of the form $S^{-1}I$ for $I$ an ideal of $\mathbb{Z}$. Therefore, $S^{-1}R$ is a PID. 5 points
3. By Prop. 6.3.9 (4) there is a one to one correspondence between the primes of $\mathbb{Z}$ not meeting $S$ (i.e., the prime ideals fully contained in the set $(2) \cup (3)$) and the prime ideals of $S^{-1}\mathbb{Z}$. In particular, the prime ideals are $0, \left(\frac{2}{1}\right), \left(\frac{3}{1}\right)$. 5 points, -1 point for forgetting 0
4. We note that all non-zero prime ideals of $S^{-1}R$ are maximal (this is even true in $\mathbb{Z}$). By Prop. 7.5.3. and Prop. 7.5.7. this means that a non-zero ideal $I$ in $S^{-1}\mathbb{Z}$ is primary if and only if $\sqrt{I}$ is prime. We note that $\sqrt{I} = 0$ iff $I = 0$. Therefore, the primary ideals are $0, \left(\frac{2^k}{1}\right), \left(\frac{3^k}{1}\right)$ for all $k \geq 1$. 5 points

□

**Exercise 3** (20 points)**.**
For a prime $p > 2$, set $R = \mathbb{F}_p[x]\Big/(x^p)$. Compute for every integer $i \geq 0$ the $R$-module

$$\mathrm{Ext}^i_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right),$$

where $\mathbb{F}_p[x]\Big/(x^j)$ is endowed with an $R$-module structure via the natural surjection $\mathbb{F}_p[x]\Big/(x^p) \twoheadrightarrow \mathbb{F}_p[x]\Big/(x^j)$ for $j = 1, 2$.

*Proof.* We construct a projective resolution of $\mathbb{F}_p[x]/(x)$ as follow; the first term is the natural surjection $\mathbb{F}_p[x]/x^p \to \mathbb{F}_p[x]/(x)$ with kernel equal to the image of the multiplication by $x$ map, $m_x : \mathbb{F}_p[x]/x^p \to \mathbb{F}_p[x]/x^p$. The kernel of $m_x$ is the image of $m_x^{p-1} : \mathbb{F}_p[x]/x^p \to \mathbb{F}_p[x]/x^p$ which has kernel equal to the image of $m_x$. In particular we get the following infinite projective resolution:

$$\ldots \xrightarrow{m_x} \mathbb{F}_p[x]/x^p \xrightarrow{m_x^{p-1}} \mathbb{F}_p[x]/x^p \xrightarrow{m_x} \mathbb{F}_p[x]/x^p$$

We compute $\mathrm{Ext}^i_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right)$ as the cohomology in degree $i$ of the following cocomplex

$$\ldots \xleftarrow[\circ m_x]{} \mathrm{Hom}(\mathbb{F}_p[x]/x^p, \mathbb{F}_p[x]/x^2) \xleftarrow{\circ m_{x^{p-1}}} \mathrm{Hom}(\mathbb{F}_p[x]/x^p, \mathbb{F}_p[x]/x^2) \xleftarrow[\circ m_x]{} \mathrm{Hom}(\mathbb{F}_p[x]/x^p, \mathbb{F}_p[x]/x^2)$$

Correct projective resolution gives 8 points

Recall that for any $R$ module $M$ there is a canonical isomorphism of $R$ modules $\mathrm{Hom}_R(R, M) \cong M$ defined by $f \to f(1)$. Moreover, since $p > 2$, we have $p - 1 \geq 2$ and so for any $f \in \mathrm{Hom}(\mathbb{F}_p[x]/x^p, \mathbb{F}_p[x]/x^2)$ the compositin $f \circ m_{x^{p-1}}(1) = x^{p-1}f(1) = 0$, in particular $\circ m_{x^{p-1}}$ is the zero morphism. The above sequence is by these observations therefore canonically isomorphic (via the isomorphism $f \to f(1)$) to the cocomplex:

$$\ldots \xleftarrow[m_x]{} \mathbb{F}_p[x]/x^2 \xleftarrow[0]{} \mathbb{F}_p[x]/x^2 \xleftarrow[m_x]{} \mathbb{F}_p[x]/x^2$$

Correct formulation of kernels and co-kernels gives additionally 6 points

We therefore have for $i$ odd:

$$\mathrm{Ext}^i_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right) \cong \mathbb{F}_p[x]/x^2/x\mathbb{F}_p[x]/x^2 \cong \mathbb{F}_p[x]/x \cong \mathbb{F}_p$$

and for $i$ even

$$\mathrm{Ext}^i_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right) = \ker(m_x) = x\mathbb{F}_p[x]/x^2 \cong \mathbb{F}_p.$$

To conclude, for all $i > 0$ we have $\mathrm{Ext}^i_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right) = \mathbb{F}_p$.

A correct conclusion for $i > 0$ gives additionally 4 points

For $i = 0$ we have by Proposition 4.3.8 that

$$\mathrm{Ext}^0_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right) = \mathrm{Hom}_R\left(\mathbb{F}_p[x]\Big/(x), \mathbb{F}_p[x]\Big/(x^2)\right).$$

Which is equal to $\ker(m_x) = x\mathbb{F}_p[x]/x^2 \cong \mathbb{F}_p$. A correct calculation of $i = 0$ gives 2 points

□

**Exercise 4** (20 points). 1. For a field $k$, we define $R = k[\![x]\!]$ as we defined $\mathbb{F}_q[\![x]\!]$ in Exercise 4 of Sheet 9. That is, $k[\![x]\!]$ is the set of formal power series $\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in k$, and addition and multiplication goes as it goes for polynomials. That is:

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) + \left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} a_i b_{j-i}\right) x^i$$

Show then that $R$ is a PID. Describe the units of $R$ and the prime elements of $R$.
10 points

2. Find a direct sum $M$ of free cyclic $R$-modules and cyclic $R$-modules with prime power annihilators such that $M$ is isomorphic as an $R$-module to the quotient module

$$R \oplus R \oplus R \Big/ R \cdot (1+x, 1, x) + R \cdot (x, x^3, x^4)$$

$$= R \oplus R \oplus R \Big/ \left\{ r \cdot (1+x, 1, x) + s \cdot (x, x^3, x^4) \;\middle|\; r, s \in R \right\}$$

10 points

*Proof.* 1. The solution of Exercise 4 of Sheet 9, part a) translates word by word to this setting. I.e., let $f = a_0 + \sum_{n>0} a_n x^n$ where $a_0 \neq 0$ define $f^{-1} = \sum_n b_n x^n$ where $b_0 = \frac{1}{a_0}$ and $b_n = -\frac{1}{a_0} \sum_{i=1}^{n} a_i b_{n-i}$ for $n \geq 1$. This shows that all $f \in k[\![x]\!] - (x)$ are invertible.
A correct identification of the units gives 2 points
Now let $I \subset k[\![x]\!]$ be a non-trivial proper ideal, then $I \subset (x)$ by what has just been said. Every $e \in I$ is of the form $x^s r$ for some $r \in R - (x)$ (since we can wrire $e = \sum_{i=s}^{\infty} a_i x^i$ and $a_s \neq 0$). Let $s$ be minimal with respect to all such integers, then clearly $I \subset (x^s)$. We want to show $x^s \in I$. By assumption there exists an $r \notin (x)$ such that $rx^s \in I$, but all such $r$ are invertible and hence $r^{-1} r x^s \in I$. In conclusion every non-zero proper ideal is of the form $(x^s)$ for some integer $s \geq 1$.
A correct argument showing $R$ is a pid gives 7 points
If $s \geq 2$ this ideal is not prime since $x \notin (x^s)$. Therefore, the only prime elements are 0 and $x$.
A correct identification of the prime elements gives 1 point

2. We are interested in the cokernel of the $R$-linear map

$$R \oplus R \longrightarrow R \oplus R \oplus R$$

defined by

$$(r, s) \to (r, s) \begin{pmatrix} 1+x & 1 & x \\ x & x^3 & x^4 \end{pmatrix}.$$

In order to solve the exercise we put the matrix into Smith normal form. In the first step we switch the first and second column, by elementary row and column operations we get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x - x^3(1+x) & 0 \end{pmatrix}$$

A correct computation of the Smith normal form gives 7 points (a minor miscalculation in a otherwise correct set-up gives a subtraction of 2 points)

In particular, there are generators $f_1, f_2, f_3$ that satisfies the relations

$$f_1 = 0, (x - x^3 - x^4) f_2 = 0.$$

Therefore,

$$M \cong k[\![x]\!]/(x - x^3 - x^4) \bigoplus k[\![x]\!] \cong k[\![x]\!]/(x) \bigoplus k[\![x]\!]$$

where the last equality comes from the fact that $1 - x^2 - x^3$ is invertible by the previous exercise. A correct conclusion gives additionally 3 points (if the generator $f_3$ is forgotten then 2 points are subtracted)

□

**Exercise 5** (20 points). 1. Show that if $0 \neq h \in \mathbb{C}[x,y]$ is a prime element, then $\mathbb{C}[x,y]\Big/(h)$ is not a field. 8 points

2. Show that if $f \in R$ is a (non-unit and non-zero) prime element of a Noetherian domain, then the only prime ideal properly contained in $(f)$ is $(0)$ (in particular the height of $(f)$ is 1). 6 points

3. Show that if $p \subseteq R$ is a height 1 prime ideal in a Noetherian UFD, then $p = (g)$, where $g \in R$ is a prime element. 6 points

*Proof.* 1. By Theorem 5.1.12 in the course notes it is sufficient to show that

$$\operatorname{trdeg}\left(\mathbb{C}[x,y]\Big/(h)\right) > 0.$$

In order to derive a contradiction suppose that the transcendence degree is zero, then $\mathbb{C}[x,y]\Big/(h) = \mathbb{C}$ since $\mathbb{C}$ is algebraically closed. In particular, this implies that there exists $\lambda_x, \lambda_y \in \mathbb{C}$ such that $x - \lambda_x \in (h)$ and $y - \lambda_y \in h$. In particular, this means that $h | x - \lambda_x$ and $h | y - \lambda_y$ but this is impossible unless $h$ is a unit, a contradiction. A correct solution gives 8 points, a promising but not complete solution gives 5 points, a promising start of a solution gives 3 points

2. In order to arrive at a contradiction let $p \subsetneq (f)$ be a non-zero prime ideal.

Let $0 \neq g \in p$, then $g \in f$ hence there exists some $r \in R$ such that $g = rf$. Since $p$ is prime either $r \in p$ or $f \in p$. If $f \in p$ then $p = (f)$. Therefore, wlog we may assume $r \in p$. Then $r \in (f)$ and so there exists $r_1$ such that $r = r_1 f$. As before w.l.o.g $r_1 \in p$. In particular there is an increasing sequence $(r) \subset (r_1) \subset (r_2) \subset \dots$. This sequence has to stabilize at some step $n$ since $R$ is Noetherian. This means that $ur_n = r_{n-1}$ for some unit $u \in R$. By construction this means that $ur_n = r_n f$, since $R$ is a domain this implies that $f = u$ is a unit. This is a contradiction, hence $(f) = p$.

A correct solution gives 6 points, an almost complete solution gives 4 points, correct ideas gives 2 points

3. We have 0 is of height zero. Let $p$ be of hight one, then $p \neq 0$. Let $f \in p$ since $R$ is a UFD there exists unique irreducible prime elements $p_i$ and unique integers $\alpha_i$ such that $g = u p_1^{\alpha_1} \dots p_n^{\alpha_n}$ for some unit $u$. Since $p$ is prime there exists some $i$ such that $p_i \in (p)$. Then $0 \subsetneq (p_i) \subset (p)$ implies that $p_i = p$ since $p$ is of hight one.

A correct solution gives 6 points, an almost complete solution gives 4 points

□