Rings and Modules – Final exam

21.01.2020, 12:15-15:15

Your Name ______________________________

This examination booklet contains 6 problems on 20 pages of paper including the front cover and the empty pages.

Please, follow the instructions below!

(1) First sign the booklet on the line provided above!
(2) Calculators, books, notes, electronic devices etc. are NOT allowed.
(3) Please, mute your phone and leave it in your bag at the back of the classroom.
(4) Do all of your work in this booklet. If you need extra paper, ask the proctors to give you yellow paper. Make sure to number the yellow pages in a clear way, so that the graders cannot get confused with the correct order of the pages.
(5) You should fully justify/explain your answers. In each question, it is always stated what results you can assume without proving. Prove all relevant computations and claims that you make.
(6) The exercises do not require any involved computations or elaborate discussions – try to be coincise.
(7) You may unstaple the booklet, we are prepared to staple it back. However, it is your responsibility to put the papers in the right order.

This booklet is divided into 3 parts: Part A, Part B, Part C.
Each part contains 2 questions in total.
For each of Part A, Part B, Part C, you should choose exactly one of the two questions and solve that.
All questions carry equal weight.

In the table below, for each of the 3 parts report in the second column (the one labelled "Question #") the number corresponding to the question that you have attempted.
Only those questions whose number is reported in the table will be marked.

Part	Question #	Maximum score	Your score	
A		25		
B		25		
C		25		
	Exam	75		=E
	Homework	210		=HW
	Total	100		$=E + \frac{25}{210} * HW$

Part A

Choose one of the two following questions and solve it. Do not forget to report on the first page of the booklet which of the two questions you solved to Part A.

Question A.1 [25pt]

Let F be a field. For $n \in \mathbb{N}_{>0}$, we shall denote

$$R_n := F[X_1, \dots, X_n],\ R'_n := F[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}].$$

(1) Prove the following statement:

Let $\mathfrak{m} \subset R_n$ be a maximal ideal. Then the field $k = R_n/\mathfrak{m}$ is an algebraic extension of F.
[*You may assume here any result proved in the lectures, if clearly stated.*] [5pt]

(2) State and prove the weak Nullstellensatz. [5pt]

(3) For any n, describe the maximal ideals of R'_n, when $F = \mathbb{R}$.
[*You should indicate generators for each maximal ideal of R'_n.*] [5pt]

(4) Assume that F is algebraically closed.
Compute the Krull dimension of R'_n. [5pt]

(5) Assume that F is algebraically closed.
Show that any prime ideal $\mathfrak{p} \subset R'_n$ is the intersection of all maximal ideals of R'_n containing $\mathfrak{p}$. [5pt]

Question A.2 [25pt]

Let F be an algebraically closed field Let n, p, q be positive integers, with $p > 1, n, q > 0$.

(1) State the definition of an integral extension of rings and the definition of an integrally closed ring. [4pt]
(2) State Noether's Normalization Theorem. [4pt]
(3) Let R be the ring $R := F[X, Y, T_1, \ldots, T_n]/(X^p Y^q - f(T_1, \ldots, T_n))$, where $f \in F[T_1, \ldots, T_n]$ is a non-constant polynomial.
Construct an integral extension $S \subset R$ with $S = F[Z_1, \ldots, Z_l]$ as guaranteed by Noether's Normalization Theorem.
What is the meaning of the integer l? [9pt]
(4) Show that in each entry of the list below the ring R is a domain and compute the integral closure:
(a) $R = F[x, y]/(x^2 - y^3)$
(b) $R = F[x, y, z]/(x^2 - yz^2)$
(c) $R = F[x, y, z]/(x^2 - yz)$
[8pt]

Part B

Choose one of the two following questions and solve it. Do not forget to report on the first page of the booklet which of the two questions you solved to Part B.

Question B.1 [25pt]

Let R be a commutative ring with unit. Let n, q, k be positive integers. We denote by $\mathrm{Mat}_k(R)$ the free R-module of $k \times k$ matrices with coefficients in R.

(1) State the Fundamental Theorem of PIDs and the Smith normal form reduction theorem. [5pt]

(2) For the following matrices, either compute, when possible, their Smith normal form or explain why they cannot be reduced to Smith Normal form:

(a) $A \in \mathrm{Mat}_n(\mathbb{R}[X])$ and all entries of A are equal to the polynomial

$$f(X) = 2020X^{2019} + 2019X^{2018} + 2018X^{2017} + 2017X^{2016} + \cdots + 3X^2 + 2X + 1.$$

(b) $B = \begin{pmatrix} 2 & 0 \\ X & 0 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}[X])$

(c) C is the same matrix as in (a), but this time you should consider it as a matrix with coefficients in $\mathbb{Z}[X]$.

[10pt]

(3) Give an example of a $\mathbb{Z}$-module M which is not finitely generated and for which there exists an endomorphism $f\colon M \to M$ which cannot be put in Smith normal form. [5pt]

(4) Let $R = \mathbb{Q}$ and let $f_{\min}(X) = (X-1)^2(X^3-3)$, $g(X) = (X-1)^4(X^3-3)$. How many different conjugation classes are there in $\mathrm{Mat}_7(R)$ of matrices with minimal polynomial $f_{\min}(X)$ and characteristic polynomial $g(X)$?

[*Recall for an endomorphism $\psi\colon V \to V$ of a $\mathbb{Q}$-vector space the minimal polynomial of ψ is the monic generator of the kernel of the ring homorphism*

$$\begin{aligned} \mathbb{Q}[X] &\to \mathrm{End}_{\mathbb{Q}}(V) \\ 0 &\mapsto 0 \\ 1 &\mapsto 1 \\ X &\mapsto \psi. \end{aligned}$$

] [5pt]

Question B.2 [25pt]

Let R be a commutative ring with unit.

(1) Define the notion of a primary ideal $I \subset R$.
Show that the radical ideal $\sqrt{I}$ of a primary ideal $I \subset R$ is prime.
State the theorem on the existence of minimal primary decompositions for ideals in a noetherian ring. What can be said in regards to the uniqueness such decompositions? [5pt]

(2) Let R be a PID. Show that R is Noetherian.
Is R Artinian? Either show that the stament holds or provide a counterexample.
[*For the first part of the question you cannot use the implication* PID $\Rightarrow$ UFD.] [4pt]

(3) Let R be a PID. Let $f \in R$ be a non-zero element and define $R' = R/(f)$.
Provide a justified answer to the following questions:
 (a) Characterize those $f \in R$ for which R' is a domain.
 (b) Is R' Artinian?
 (c) Compute the radical ideal $\sqrt{0} \subset R'$.
Answer questions (a-d) when R is a UFD instead of a PID. [4pt]

(4) Let R be a UFD and let $I \subset R$ be a principal ideal.
Show that there exists a unique minimal primary decomposition of I.
Show that the uniqueness does not hold if I is not assumed to be principal. [5pt]

(5) Show that if $I = \sqrt{I}$ is a radical ideal and $ab \in I$, then $I = \sqrt{I + (a)} \cap \sqrt{I + (b)}$. [7pt]

Part C

Choose one of the two following questions and solve it. Do not forget to report on the first page of the booklet which of the two questions you solved to Part C.

In this section you can assume the following result:
Let Q be an R-module. If $Q_{\mathfrak{m}} = 0$, for any maximal ideal $\mathfrak{m} \subset R$, then $Q = 0$.

Question C.1 [25pt]

Let R be a commutative ring with unit.

(1) Define what it means for a subset $T \subset R$ to be a multiplicatively closed subset of R. Define the localization $T^{-1}R$ of R at T and the natural homomorphism $R \to T^{-1}R$. [6pt]

(2) Let $\mathfrak{p} \subset R$ be a prime ideal. Define the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$. When is the natural homomorphism $R \to R_{\mathfrak{p}}$ injective? [5pt]

(3) Show that if for every prime $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ contains no nilpotent element then R contains no nilpotent element. [7pt]

(4) Assume that R is a domain. Let M be an R-module.
recall that the submodule $\mathrm{Tor}(M) \subset M$ of torsion elements of M is defined as

$$\mathrm{Tor}(M) = \{m \in M \mid \exists r \in R \setminus \{0\} \text{ such that } rm = 0\}.$$

Let S be a multiplicatively closed subset of R. Show that $\mathrm{Tor}(S^{-1}M) = S^{-1}(\mathrm{Tor}(M))$, where $S^{-1}(\mathrm{Tor}(M))$ denotes the submodule generated by the image of $\mathrm{Tor}(M)$ in $S^{-1}M$.
Show that for an R-module M the following are equivalent:
   (a) M is torsion-free;
   (b) $M_{\mathfrak{p}}$ is torsion-free, for all prime ideals $\mathfrak{p} \subset R$;
   (c) $M_{\mathfrak{m}}$ is torsion-free, for all maximal ideals $\mathfrak{m} \subset R$;

[7pt]

QUESTION C.2 [25PT]

Let R be a commutative ring with unit. Let M, N be R-modules.

(1) Define the R-modules $\mathrm{Ext}^i_R(M, N)$. [5pt]

(2) Construct a projective resolution for an R-module M

$$\mathcal{P}\colon\ \cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_j is a free R-module.
Moreover, show that if R, M are Noetherian, then each P_i can be taken to be a free and finitely generated R-module. [5pt]

(3) Let $\mathfrak{p} \subset R$ be a prime ideal. Let P be a finitely generated free R-module. Show that there is an isomorphism or $R_\mathfrak{p}$-modules

$$\phi_{P,N}\colon \mathrm{Hom}_R(P, N)_\mathfrak{p} \to \mathrm{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, N_\mathfrak{p}),$$

where the module $\mathrm{Hom}_R(P, N)_\mathfrak{p}$ is the localization of the R-module $\mathrm{Hom}_R(P, N)$ at $\mathfrak{p}$.
[*Hint: Use the universal property of free modules.*] [5pt]

(4) Let P_1, P_2 be free R-modules and let $f\colon P_1 \to P_2$ be a homomorphism of R-modules.
Show that f induces a homorphism of $R_\mathfrak{p}$-modules $f_\mathfrak{p}\colon (P_1)_\mathfrak{p} \to (P_2)_\mathfrak{p}$.
Deduce that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_R(P_2, N)_\mathfrak{p} \arrow[r, "\mathrm{Hom}_R(f,N)_\mathfrak{p}"] \arrow[d, "\phi_{P_2,N}"] & \mathrm{Hom}_R(P_1, N)_\mathfrak{p} \arrow[d, "\phi_{P_1,N}"] \\
\mathrm{Hom}_{R_\mathfrak{p}}((P_2)_\mathfrak{p}, N_\mathfrak{p}) \arrow[r, "\mathrm{Hom}_{R_\mathfrak{p}}(f_\mathfrak{p},N_\mathfrak{p})"] & \mathrm{Hom}_{R_\mathfrak{p}}((P_1)_\mathfrak{p}, N_\mathfrak{p}),
\end{tikzcd}$$

where $\mathrm{Hom}_R(f, N)_\mathfrak{p}$ is the localization of the morphism $\mathrm{Hom}_R(f, N)$ at $\mathfrak{p}$. [5pt]

(5) Use $(2-4)$ to show that there exists an isomorphism

$$\psi\colon \mathrm{Ext}^i_R(M, N)_\mathfrak{p} \to \mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}).$$

[5pt]
