Rings and Modules – Final exam

30.01.2018, 8:15-11:15

Your Name ______________________________

This examination booklet contains 6 problems on 20 pages of paper including the front cover and the empty pages.

First sign the booklet above! Do all of your work in this booklet, if you need extra paper, ask the proctors to give you yellow paper, show all relevant computations and justify/explain your answers. The exercises do not require any involved computations or elaborate discussions – try being to the point. Calculators, books, notes, electronic devices etc. are NOT allowed. In particular, please mute the ringer and leave the phone in you bag. You might unstaple the booklet, we are prepared to staple it back. However, it is your responsibility to put the papers in the right order.

Problem	Possible score	Your score
1	13	
2	15	
3	20	
4	20	
5	20	
6	12	
Total	100	

QUESTION 1 [13]

Let R be an integral domain, and let K be its fraction field. Prove the following statements:

(1) if $f \in R$ is a non-zero element, then $\mathrm{Ext}^1_R(R/(f), K) = 0$, [7]

(2) more generally if $f_1, \dots, f_n$ is a sequence of elements such that for every $1 \leq i \leq n$ the multiplication by f_i is injective on $R/(f_1, \dots, f_{i-1})$ then

$$\mathrm{Ext}^1_R(R/(f_1, \dots, f_n), K) = 0.$$

[6]

Question 2 [15]

Let k be a field. We set

$$R = k[x, y, z, u]/(xyzu - 1).$$

Recalling the proof of Noether's normalization find an integral extension $S \subset R$ such that S is a polynomial ring in n variables. What is n equal to? [8]

Next, state the going up theorem and use it for the proof of the fact that Krull dimension $\dim R = n$ (you are required to use going up for this). [7]

QUESTION 3 [20]

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Formulate Nakayama's lemma for R, and then show that every finitely generated flat module F over R is free by proving the following statements. [5]

(1) A homomorphism $M \to N$ of finitely generated R-modules is surjective if and only if the induced map of $R/\mathfrak{m}$ vector spaces $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective, [5]

(2) Let $f_1, \ldots, f_n$ be elements of F such that their images $\overline{f_1}, \ldots, \overline{f_n}$ under the natural map $F \to F/\mathfrak{m}F$ form a basis. Prove that the map $g\colon R^n \to F$ defined by the associations $e_i \mapsto f_i$ is surjective. [5]

(3) Prove that the kernel $K = \{x \in R^n : g(x) = 0\}$ is zero, and hence g is an isomorphism, by considering the exact sequence:

$$0 \to K \to R^n \to F \to 0$$

and the associated long exact sequence of $\operatorname{Tor}^i_R(-, R/\mathfrak{m})$ modules. You may use Nakayama's lemma and the fact that an R-module M is flat if and only if $\operatorname{Tor}^1_R(M, P) = 0$ for every R-module P. [5]

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1. **Question 4** [20]

Let A be a noetherian ring, and let $I = \sqrt{0}$ be its nilpotent radical.

(1) Recall the definition of the Krull dimension, and then prove that the Krull dimensions of the rings A and A/I are equal. In particular, for every ideal J in $k[x_1, \ldots, x_n]$ the dimension of $k[x_1, \ldots, x_n]/J$ equals the dimension of $k[x_1, \ldots, x_n]/\sqrt{J}$. [4]

Let k be a field. For each of the following rings R: compute the nilpotent radical of R, compute the prime ideals of height zero of R, compute the Krull dimension of R.

(2) $R = k[x, y, z]/(xz^3, yz^2)$. [8]
(3) $R = k[x, y, z]/(x^6 + y^6 + z^6)$ (this depends on the characteristic of k). [8]

For (2) you may use the fact that a prime ideal $\mathfrak{p}$ contains an intersection of ideals $I \cap J$ if and only if it contains either I or J.

Recall that the characteristic of the field k is either zero or the smallest prime number p such that $p = 0$ in k. Note that if characteristic is equal to $p > 0$ then $a^p + b^p = (a + b)^p$ for every $a, b \in k$.

Question 5 [20]

Prove the following statements.

(1) Let $A \subset B$ be an inclusion of commutative integral domains, and let C be the integral closure of A inside B. Let S be a multiplicately closed set in A. Prove that $S^{-1}C$ is the integral closure of $S^{-1}A$ inside $S^{-1}B$. [6]

(2) Deduce that if A is an integrally closed domain then $S^{-1}A$ is also integrally closed. [4]

(3) Prove that the ring

$$R = k[x, y, z]/\left((x+1)^4 - z(y + z^{2019})^4\right).$$

is a domain. Compute its integral closure, and then find an element $u \in R$ such that the localization R_u is integrally closed. [10]

Question 6 [12]

State and prove Hilbert's Basis Theorem.
