EPFL- Section de Mathématiques Fall 2017

Rings and modules – Final

18.01.2018, 12:15-15:15

Your Name ______________________________

This examination booklet contains 8 problems on 28 sheets of paper including the front cover and the empty sheets.

Do all of your work in this booklet, if you need extra paper, ask the proctors to give you yellow paper, show all your computations and justify/explain your answers. Calculators, books, notes, electronic devices etc. are NOT allowed.

Problem	Possible score	Your score
1	10	
2	12	
3	10	
4	5	
5	18	
6	20	
7	15	
8	10	
Total	100	

QUESTION 1 [10]

Compute $\mathrm{Tor}_i^{\mathbb{Z}}((\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(4,6), \mathbb{Z}/m\mathbb{Z})$ for all $i \geq 0$ and all $m \in \mathbb{Z}$.

Question 2 [12]

Let R be a commutative ring which is an integral domain. Let M be a finitely generated R-module. Let $Tor(M)$ be the torsion submodule of M.

(a) Show that $\mathrm{Tor}_1^R(R/rR, M)$ is the r-torsion submodule of M for all $r \in R$. [6]
(b) Show that if M is flat then $Tor(M) = 0$. [3]
(c) Show that if R is a PID, then M is flat if and only if $Tor(M) = 0$. [3]

Question 3 [10]

Let e_1, e_2, e_3 be a basis of the free $\mathbb{Z}$-module $\mathbb{Z}^3$. Consider the submodule N generated by $7e_1 - 3e_2$, $3e_1 - e_2 + 4e_3$, $6e_1 - 2e_2 + 2e_3$. Compute the Smith normal form of the following matrix with entries in $\mathbb{Z}$:

$$A = \begin{pmatrix} 7 & 3 & 6 \\ -3 & -1 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

Then put the $\mathbb{Z}$-module $M = \mathbb{Z}^3/N$ into the form described by the structure theorem for finitely generated modules over a PID.

Question 4 [5]

Let R be a commutative ring. Let T be a multiplicatively closed subset of R. Show that if R is noetherian, then $T^{-1}R$ is noetherian.

Question 5 [18]

Let k be a field. For each of the following rings R compute the nilpotent radical of R, compute a minimal primary decomposition of (0), compute the prime ideals of height 0 of R, compute the Krull dimension of R.

(a) $R = k[x, y]/(x^2 y, xy^2)$. [6]
(b) $R = k[x, y, z]/(x^4 + y^4 + zy)$. [6]
(c) $R = k[x]/(x^2 + 1)$ (the answer should depend on the field). [6]

Question 6 [20]

(a) Let R be a commutative integral domain. Let $T \subseteq R$ be a multiplicatively closed subset. Prove that if R is integrally closed, then $T^{-1}R$ is integrally closed. [5]

(b) Show that the following rings are integral domains and compute their integral closure:

(i) $\mathbb{R}[x, y]/(xy - 1)$ [5]

(ii) $\mathbb{Q}[x, y, z]/((x + y)^2 - yz^4)$ [10]

Question 7 [15]

Let R be a commutative ring containing a multiplicatively closed subset T. Endow the R-module $R \oplus R$ with the coordinatewise multiplication $(r_1, r_2) \cdot (r_1', r_2') := (r_1 r_1', r_2 r_2')$ for all $r_1, r_1', r_2, r_2' \in R$. Show that the subset $T \oplus T \subseteq R \oplus R$ is a multiplicatively closed subset of $R \oplus R$. Then prove, using the universal property of localisation, that there is an isomorphism of rings $(T \oplus T)^{-1}(R \oplus R) \cong (T^{-1}R) \oplus (T^{-1}R)$.

QUESTION 8 [10]

State and prove the Going-Up Theorem. You can use without proof all the preliminary lemmas and propositions we proved before the actual proof of the Going-Up Theorem.
