Rings and modules – Final

30.01.2017, 8:15-11:15

Your Name ______________________________

This examination booklet contains 7 problems on 24 sheets of paper including the front cover and the empty sheets.

Do all of your work in this booklet, if you need extra paper, ask the proctors to give you yellow paper, show all your computations and justify/explain your answers. Calculators, books, notes, electronic devices etc. are NOT allowed.

Problem	Possible score	Your score
1	30	
2	5	
3	20	
4	10	
5	10	
6	10	
7	15	
Total	100	

By k we always denote an arbitrary field.

Question 1 [30]

For each of the following rings determine the Krull dimension, the nilradical and one minimal primary decomposition of (0).

(a) $R = k[x,y]/(x^2,y^2)$. In this case also consider the following module $M := R/(\overline{x}) \oplus R/(\overline{y})$, where the bar denotes the residue classes of the corresponding elements. Compute $\dim_k M$ and $\dim_k R$. Is M isomorphic to R? [12]
(b) $R = k[x,y]/(x^3, xy^2)$. [6]
(c) $R = k[x,y]/(x^3 + (y^2+y)x + y)$. [6]
(d) $R = k[x,y,z]/(x^2+y^2+z^2)$ (your answer should depend on the characteristic of k). [6]

Throughout this solution, we denote elements of polynomial rings by x, y, z and 0, and their images in the quotients by $\overline{x}$,$\overline{y}$, $\overline{z}$ and $\overline{0}$.

(a) Let $I \subset k[x,y]$ be (x^2,y^2). The radical of I satisfies $\sqrt{I} \supset (x,y) \supset I$. But the ideal (x,y) is maximal, so is radical. So $\sqrt{(\overline{0})}$ in R is $(\overline{x},\overline{y})$. The nilradical is the intersection of all prime ideals of R. Therefore as the nilradical is maximal, it is the unique prime ideal of R. So R has Krull dimension 0. (x^2,y^2) is itself primary because it's radical is maximal.
First note that any element of R can be written uniquely as $a+b\overline{x}+c\overline{y}+d\overline{xy}$ for $a,b,c,d \in k$. As a k-vector space, M has basis $\{(1,1),(1,\overline{x}),(\overline{y},1),(\overline{y},\overline{x})\}$, and so had dimension 4. R has basis $\{1,\overline{x},\overline{y},\overline{xy}\}$ and so also has dimension 4 as a k-vector space. M is not isomorphic to R as an R-module, because for example $Tor_x R = (\overline{x})$ is a one dimensional k-vector space and $\mathrm{Tor}_x M = ((1,\overline{x}),(\overline{y},\overline{x}))$ is a two dimensional k-vector space.
(b) If $I \subset k[x,y]$ is the ideal (x^3,xy^2) we have $I \subset (x) \subset \sqrt{I}$, and as (x) is prime, it is also radical, and so $\sqrt{(\overline{0})} = (\overline{x})$. $(\overline{x})$ is the intersection of all prime ideals containing I, and is itself prime. So it is the unique minimal prime ideal containing I. $k[x,y]$ is Krull dimension two, and contains the maximal chain of prime ideals $(0) \subset (x) \subset (x,y)$. Therefore $(\overline{x}) \subset (\overline{x},\overline{y})$ is a maximal chain of prime ideals in R. So R has Krull dimension 1. I can be written as $(x) \cap (x^3,y^2)$, which gives a primary decomposition. (x^3,y^2) is radical because its radical is (x,y) which is maximal. Therefore a primary decomposition of $(\overline{0})$ is $(\overline{x}) \cap (\overline{x}^3,\overline{y}^2)$.
(c) The ideal $I = (x^3+(y^2+y)x+y)$ in $k[x,y]$ is prime by Eisenstein's criterion on $k[y][x]$ applied to the irreducible polynomial y in $k[y]$. Therefore $\sqrt{(\overline{0})} = (\overline{0})$, and this is a primary decomposition. In $k[x,y]$ we have $0 \subset (x^3+(y^2+y)x+y) \subset (x,y)$, and this is a maximal chain of prime ideals as $k[x,y]$ has Krull dimension 2. Therefore $(\overline{0}) \subset (\overline{x},\overline{y})$ gives a maximal chain of primes in R and the Krull dimension of R is 1.
(d) Firstly, suppose the characteristic of k is 2. We have $(x^2+y^2+z^2) = (x+y+z)^2$. The nilradical satisfies $\sqrt{(\overline{0})} \supset (\overline{x}+\overline{y}+\overline{z}) \supset I$, and $(\overline{x}+\overline{y}+\overline{z})$ is prime. Therefore $\sqrt{(\overline{0})} = (\overline{x}+\overline{y}+\overline{z})$. We claim that the ideal $(x+y+z)^2$ is primary. For suppose $xy \in (x+y+z)^2$. Then as $k[x,y,z]$ is a UFD, either $(x+y+z)^2|f$ or $(x+y+z)|g$. In the latter case $g^2 \in (x+y+z)^2$. As the nilradical $(\overline{x}+\overline{y}+\overline{z})$ is prime, it is the minimal prime ideal containing $(\overline{0})$. We

have a chain of prime ideals $(0) \subset (x+y+z) \subset (x+2y, x+2z) \subset (x, y, z)$ in $k[x, y, z]$ (which is of Krull dimension 3). Therefore R is of Krull dimension 2.

Now suppose R has characteristic not equal to 2. Then $(x^2+y^2+z^2)$ is prime by Eisenstein: either k contains a square root of -1 and so $x^2+y^2 = (x+iy)(x-iy)$ with $x+iy \neq x-iy$, or it does not, in which case x^2+y^2 is itself irreducible in $k[x, y]$. Therefore $(\overline{0})$ is equal to its own nilradical and is its own primary decomposition. R has Krull dimension at least 2 because it has transcendence degree 2: $\overline{x}$ and $\overline{y}$ are algebraically independent, for any relation involving them also involves z. It cannot have transcendence degree larger than 2 because $0 \subset (x^2+y^2+z^2)$ is a non-trivial chain of primes in $k[x, y, z]$.

Question 2 [5]

Let R be a (not necessarily commutative) ring. What does it mean for a module M to be simple? Prove that if $0 \neq M$ is a simple R-module then

$$End_R(M) = \{\phi : M \to M | \phi \text{ is an } R\text{−module homomorphism}\}$$

is a skew field.

An R-module M is simple if it has no submodules other than 0 and M.

Let M be a non-zero simple module, and $\phi \in End_R(M)$ a non-zero element of $End_R(M)$. We must show that ϕ has an inverse. For this it is enough to show that it is bijective. But $\ker(\phi)$ and $\mathrm{im}(\phi)$ are both submodules of M, and neither can be zero otherwise ϕ is zero. So both are equal to M and hence ϕ is injective and surjective respectively.

QUESTION 3 [20]

Let R be a commutative ring. We say that an R-module M is Artinian if it satisfies the descending chain condition on submodules; that is if every descending chain of submodules

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

satisfies $M_i = M_{i+1}$ for all i sufficiently large.

(a) Suppose M has a submodule N. Prove that M is an Artinian R-module if and only if both N and M/N are Artinian. [10]

(b) Suppose R is Artinian as an R-module. Prove that every prime ideal of R is maximal.

[Hint: reduce it to the integral domain case by quotienting, and then show that every Artinian integral domain is a field.] [10]

(a) Firstly suppose that M is Artinian. A chain of submodules of N are also submodules of M and so they stabilise and N is Artinian. Similarly a chain of submodules of M/N correspond to a chain of submodules of M under the correspondence between submodules of M/N and submodules of M containing N. Hence M/N is also Artinian.

Conversely suppose that both N and M/N are Artinian, and that we have a chain of submodules

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

in M.

Then

$$M_1 \cap N \supseteq M_2 \cap N \supseteq M_3 \cap N \supseteq \dots$$

is a chain of submodules of N, and so for i sufficiently large, $M_i \cap N = M_{i+1} \cap N$. Replace the sequence M_i by a truncation to assume that this holds for all i. But we also have $M_i/(N \cap M_i) \cong (N + M_i)/N$. The right hand side is a chain of submodules of M/N and so stabilises. That is, $M_i/(N \cap M_i)$ stabilises for i sufficiently large. But as $N \cap M_i$ is a fixed submodule of M, this implies that M_i stabilises.

(b) Suppose that P is a prime ideal of R. R/P is Artinian by the first part, so, by the correspondence between ideals of R/P and ideals of R containing P, we may replace R by R/P to assume R is an integral domain and $P = (0)$.

Now suppose that (0) is not maximal. Then there is some non-zero ideal $I \subset R$ which is not equal to R. In particular I contains some non-zero element a. But then (a) is also a non-zero ideal, contained in I, so there is a chain of ideals

$$(a) \supset (a^2) \supset (a^3) \supset ...$$

By the Artinian property, this chain stabilises: i.e. $(a^i) = (a^{i+1})$ for some i. This means that we can write $a^i = ra^{i+1}$ for some $r \in R$. But R is an integral domain and a^i is non-zero, so this implies that $ar = 1$. But then $(a) = R$, which gives a contradiction to the existence of the ideal I.

Question 4 [10]

Let A be the following matrix with entries in $\mathbb{Z}$:

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 6 & -7 & 3 \\ 0 & 6 & 2 \end{pmatrix}$$

Put A into Smith normal form. Hence put the $\mathbb{Z}$-module M generated by m_1, m_2 and m_3 and subject to the relations

$$\begin{aligned} -2m_1 + 6m_2 &= 0 \\ 3m_1 - 7m_2 + 6m_3 &= 0 \\ -m_1 + 3m_2 + 2m_3 &= 0 \end{aligned}$$

into the form described by the structure theorem for finitely generated modules over a PID.

We put the given matrix into Smith normal form using row and column operations:

$$\begin{pmatrix} -2 & 3 & -1 \\ 6 & -7 & 3 \\ 0 & 6 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & -2 \\ 3 & -7 & 6 \\ 2 & 6 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -2 \\ -3 & -7 & 6 \\ -2 & 6 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 0 \\ 0 & 12 & -4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 12 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Let $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the linear map given by A. The image of f is equal to the kernel of the natural projection $\mathbb{Z}^3 \to M$. By changing bases of the two copies of $\mathbb{Z}^3$ so that the matrix of M is in Smith normal form, we have found generators of the module M which exhibit it as

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$

Question 5 [10]

Let R be a commutative ring containing a multiplicative subset T. Prove, using the universal property of localisation, that $T^{-1}(R[x]) \cong (T^{-1}R)[x]$.

We show that $(T^{-1}R)[x]$ satisfies the universal property required of $T^{-1}(R[x])$. To this end, suppose that there is a ring homomorphism $f : R[x] \to S$ such that the image of every element of T is invertible.

This in particular gives a homomorphism $f_R : R \to S$ by restricting to the constants in $R[x]$. The universal property of $T^{-1}R$ gives a unique homomorphism $g_R : T^{-1}R \to S$. To give a homomorphism $g : (T^{-1}R)[x] \to S$, it is enough to give such a homomorphism g_R and also specify the image of x, so we define g to extend g_R and satisfy $g(x) = f(x)$. Define $i_R : R \to T^{-1}(R)$ to be the natural inclusion, and define $i : R[x] \to (T^{-1}R)[x]$ to be the natural extension of i_R to $R[x]$. These homomorphisms satisfy $f = g \circ i$. Suppose there is another g' which satisfies $f = g' \circ i$. Let g'_R the restriction of g' to the constant polynomials. By the universal property of $T^{-1}R$, $g_R = g'_R$. But then g and g' agree both on $T^{-1}R$ and at x, and so they are equal as homomorphisms $(T^{-1}R)[x] \to S$. This completes the universal property of $T^{-1}(R[x])$.

Question 6 [10]

State and prove the Noether normalisation theorem for the case of infinite base fields. You can use without proof all the preliminary lemmas and propositions we proved before the actual proof of Noether normalisation.

Let F be an infinite field, and R is a quotient of $F[x_1, \ldots, x_n]$. Then there is subring S of R such that $S \cong F[t_1, \ldots, t_r]$ and R is integral over S.

Proof:

Let $\overline{x}_i$ be the residue classes of x_i in R. Then may assume that $\overline{x}_1, \ldots, \overline{x}_r$ are algebraically independent, and $\overline{x}_{r+1}, \ldots, \overline{x}_n$ are algebraic over $F(\overline{x}_1, \ldots, \overline{x}_r) \cong F(t_1, \ldots, t_r)$.

From here we proceed by induction on $n-r$. Since $\overline{x}_n$ is algebraic over $F(\overline{x}_1, \ldots, \overline{x}_r)$, it is also algebraic over $F(\overline{x}_1, \ldots, \overline{x}_{n-1})$. So, there is a polynomial $g(y_1, \ldots, y_n) \in F[y_1, \ldots, y_n]$ such that $g(\overline{x}_1, \ldots, \overline{x}_n) = 0$, and $g(\overline{x}_1, \ldots, \overline{x}_{n-1}, y_n) \neq 0$ as a polynomial in y_n. Let G be the highest degree homogeneous part of g, and let d be its degree. Then, as F is infinite, we may choose c_i, such that $F(c_1, \ldots, c_{n-1}, 1) \neq 0$. set $\tilde{y}_i = y_i - c_i y_n$ for $1 \leq i \leq n-1$. Consider the equation
(1)
$0 = g(y_1, \ldots, y_n) = g((y_1 - c_1 Y_n) + c_1 y_n, \ldots, (y_{n-1} - c_{n-1} Y_n) + c_{n-1} y_n, y_n) = g(\tilde{y}_1 + c_1 y_n, \ldots, \tilde{y}_2 + c_2 y_n, y_n)$
Consider the latter polynomial as a polynomial in y_n. Then this polynomial has degree at most d, and the coefficient of the y_n^d monomial is $F \ni G(c_1, \ldots, c_{n-1}, 1) \neq 0$. Hence, dividing by this element of F yields a monic polynomial in y_n.

So, $\overline{x}_n$ is integral over $R' := F(\overline{x}_1 - c_1 \overline{x}_n, \ldots, \overline{x}_{n-1} - c_n \overline{x}_n)$, which is the quotient of $F[x_1 - c_1 x_n, \ldots, x_{n-1} - c_n x_n] \subseteq F[x_1, \ldots, x_{n-1}]$.

Question 7 [15]

(a) Let R be a commutative integral domain. Prove that if R is a UFD then it is integrally closed (in $\mathrm{Frac}(R)$). [5]

(b) Let $R = k[x, y, z]/(y^3 + y^2x^2 + yx^2 + x^3z)$. You may assume without proof that R is an integral domain. Compute the integral closure of R. [10]

(a) Suppose R is not integrally closed. Suppose $\frac{a}{b} \in \mathrm{Frac}(R)$ and that there is a monic polynomial $f(t) = t^n + \sum_{i=0}^{n-1} a_i t^i$ with coefficients in R such that $f(\frac{a}{b}) = 0$. We may assume for a contradiction that $\frac{a}{b} \notin R$ and so $b \nmid a$. We can reduce the fraction and so assume that if u is an irreducible element of R and $u|b$ then $u \nmid a$. We have that $(\frac{a}{b})^n + \sum a_i(\frac{a}{b})^i = 0$ and multiplying through by b^i gives $a^n = -\sum_{i=0}^{n-1} a^i b^{n-i}$. But b divides the right hand side, and so if u is any irreducible factor of b (which exists as b is not a unit) then $u|a^n$ by unique factorisation. Thus $u|a$, which gives a contradiction, so there can be no such element $\frac{a}{b}$.

(b) First we search for elements of $\mathrm{Frac}(R)$ which satisfy a monic polynomial with coefficients in R. The equation $y^3 + y^2x^2 + yx^2 + x^3z = 0$ suggests one such: we can divide through by x^3 to get $(\frac{y}{x})^3 + (\frac{y}{x})^2 x + \frac{y}{x} + z = 0$ shows that $\frac{y}{x}$ satisfies the polynomial $t^3 + xt^2 + t + z$.

Let S be the ring obtained by adjoining $\frac{y}{x}$ to R. It is contained in the integral closure of R, and we show that in fact S is integrally closed.

By denoting $\frac{y}{x}$ by t, S is isomorphic to the ring $k[x, y, z, t]/(y - tx, t^3 + t^2x + t + z)$, which in turn is isomorphic to $k[x, t]$. This polynomial ring is a UFD so by the first part it is integrally closed.