EPFL - Fall 2024 Domenico Valloni
Rings and modules **Exercises**
Sheet 4 - Solutions

There was one bonus exercise on this problem sheet. The exercise was denoted by the symbol ♠ next to the exercise number.

Exercise 1. Let R be a commutative ring, and let M be an R-module.

(1) Show that $\mathrm{Hom}_R(M,-)$ is *left exact.* That is, for any short exact sequence of R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0 \ ,$$

there is an induced exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,N') \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N'') \ .$$

(2) Give an example of a ring R and an R-module M such that $\mathrm{Hom}_R(M,-)$ is not *right exact.* That is, give an example of a surjection of R-modules $N \twoheadrightarrow N''$ such that the induced morphism $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N'')$ is not surjective.

Proof. (1) Suppose that

$$0 \longrightarrow N' \xrightarrow{i} N \xrightarrow{s} N'' \longrightarrow 0 \ ,$$

is exact. We want to show that

$$0 \longrightarrow \mathrm{Hom}_R(M,N') \xrightarrow{i\circ -} \mathrm{Hom}_R(M,N) \xrightarrow{s\circ -} \mathrm{Hom}_R(M,N'') \ ,$$

is exact. Let $\phi \in \mathrm{Hom}_R(M,N')$ and suppose it is mapped to 0, i.e. $i \circ \phi : M \to N'$ is the zero morphism. Since i is injective this implies that $\phi = 0$. So we get exactness at $\mathrm{Hom}_R(M,N')$. To check exactness in the middle, observe that since $s \circ i = 0$ we have the containment $\mathrm{im}(i \circ -) \subset \ker(s \circ -)$. Let $\phi \in \mathrm{Hom}_R(M,N)$ be such that $s \circ \phi : M \to N''$ is the zero morphism. Then $\phi(M) \subset \ker(s) = i(N')$, and therefore ϕ factors through $i : N' \to N$.

(2) Let $R = \mathbb{Z}$. Consider the surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and let $M = \mathbb{Z}/2\mathbb{Z}$. The induced morphism

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\right) \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\right)$$

can not be surjective since the first group is zero, but the other is not.

□

Exercise 2. Let $R = k[x,y]$ where k is a field. Extend the complex below to a free resolution $F_\bullet$ of the R-module $k \cong R/(x,y)$. Then compute $\mathrm{Ext}^i_{F_\bullet}(k,R)$ for each i, and note that you get the same as for the resolutions in Example 5.3.9 in the printed course notes.

$$R \oplus R \oplus R \longrightarrow R \longrightarrow k \longrightarrow 0$$

The first morphism is defined by sending a basis to the following elements:

$$(1,0,0) \mapsto x, (0,1,0) \mapsto y, (0,0,1) \mapsto x+y$$

and the second morphism is the natural surjection $R \to k$.

[*Remark:* This is an example of the fact that the Ext-modules $\mathrm{Ext}^i_{F_\bullet}(M,N)$ don't depend on the free resolution $F_\bullet$ of M.]

Proof. The kernel of the first map is the set of those $(a,b,c) \in R^{\oplus 3}$ such that $0 = ax + by + c(x+y) = (a+c)x + (b+c)y$. As R is UFD this means that $a + c = yd$ and $b + c = -xd$ for some $d \in R$. That is, we have $a = yd - c$ and $b = -xd - c$. Equivalently $a = yd - e$ and $b = -xd - e$ and $c = e$ (where e and d are arbitrary elements of R). From here one can read off the following extension to a free resolution:

$$0 \longrightarrow R \oplus R \longrightarrow R \oplus R \oplus R \longrightarrow R \longrightarrow k \longrightarrow 0$$

$$(1,0,0) \longmapsto x$$

$$(0,1,0) \longmapsto y$$

$$(0,0,1) \longmapsto x + y$$

$$(1,0) \longmapsto (1,1,-1)$$

$$(0,1) \longmapsto (y,-x,0)$$

Upon applying $\mathrm{Hom}_R(\_, R)$ to the projective resolution determined by the complex above (removing k) and identifying $R^{\oplus n} \cong \mathrm{Hom}_R(R^{\oplus n}, R)$, we get

$$0 \longleftarrow R \oplus R \longleftarrow R \oplus R \oplus R \longleftarrow R \longleftarrow 0$$

$$(x,y,x+y) \longmapfrom 1$$

$$(1,y) \longmapfrom (1,0,0)$$

$$(1,-x) \longmapfrom (0,1,0)$$

$$(-1,0) \longmapfrom (0,0,1)$$

(Notice that on the level of matrices, the morphisms here are obtained from the morphisms above by transposing the matrix.) We calculate the cohomology of this complex, The first map is injective, hence $H^0 = 0$, i.e., $\mathrm{Ext}^0_{F_\bullet}(k,R) = 0$. The solution to the system

$$\begin{aligned} r_1 + r_2 - r_3 &= 0 \\ r_1 y - r_2 x &= 0 \end{aligned}$$

can easily seen to be $r_1 = rx, r_2 = ry, r_3 = r(x+y)$ for some $r \in R$. Therefore the above complex is exact in degree one and $\mathrm{Ext}^1_{F_\bullet}(k,R) = 0$. Finally, the image of the last map is $R \oplus (x,y)$ (because $r_1 y - r_2 x$ runs through (x,y) for r_1, r_2 running through R and we can use r_3 to get any element in the first coordinate). Thus the co-kernel is $(R \oplus R)\big/(R \oplus (x,y)) \cong R/(x,y) \cong k$. Therefore, $\mathrm{Ext}^2_{F_\bullet}(k,R) = k$. This agrees with the values for these groups given by the resolutions in Example 5.3.9 in the printed course notes. □

Exercise 3. Let $0 \to M \xrightarrow{i} Z \xrightarrow{p} N \to 0$ be a short exact sequence of R-modules.

(1) A *section* of p is a morphism $s\colon N \to Z$ such that $p \circ s = \mathrm{id}_N$. Show that p admits a section if and only if there exists an isomorphism $\Phi : M \oplus N \xrightarrow{\cong} Z$ and a commuting diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & Z & \xrightarrow{p} & N & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \Phi} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{e} & M \oplus N & \xrightarrow{\pi} & N & \longrightarrow & 0 \end{array}$$

(2) A *section* of i is a morphism $q\colon Z \to M$ such that $q \circ i = \mathrm{id}_M$. Show that i admits a section if and only if there exists an isomorphism $\Psi : Z \xrightarrow{\cong} M \oplus N$ and a commuting diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & Z & \xrightarrow{p} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \Psi} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{e} & M \oplus N & \xrightarrow{\pi} & N & \longrightarrow & 0 \end{array}$$

We say that a short exact sequence satisfying any of these conditions is split exact.

Proof. (1) Suppose that we have a commuting diagram as the one described in the exercise. Define $s : N \to Z$ by $N \xrightarrow{e_N} M \oplus N \xrightarrow{\Phi}_{\cong} Z$ where e_N is the canonical inclusion. We need to check that $p \circ s$ is equal to the identity on N. By the commutativity of the diagram $p = \pi \circ \Phi^{-1}$ and hence $p \circ s = \pi \circ \Phi^{-1} \circ \Phi \circ e_N = \pi \circ e_N = \mathrm{id}_N$.
Conversely, suppose that $s : N \to Z$ is a section of p. Define $\Phi : M \oplus N \to Z$ by $\Phi(m,n) = i(m) + s(n)$. Then for any $z \in Z$, let $n = p(z)$. Now $z - s(n)$ is in $\ker p = \mathrm{im}\, i$, so let m be a preimage under i. Then

$$\Phi(m,n) = i(m) + s(n) = z - s(n) + s(n) = z,$$

so as $z \in Z$ was arbitrary, Φ is surjective. On the other hand, if $\Phi(m,n) = 0$, then $0 = p \circ \Phi(m,n) = n$ and thus $i(m) = 0$ which also gives $m = 0$. Hence Φ is an isomorphism. As also $\Phi \circ e = i$ and $p \circ \Phi = \pi$, the diagram commutes.

(2) If the diagram exists we can define q as the composition $Z \xrightarrow{\Psi}_{\cong} M \oplus N \xrightarrow{\pi_M} M$ where π_M is the canonical projection. We need to check that $q \circ i$ is equal to the identity on M. By the commutativity of the diagram $i = \Psi^{-1} \circ e$ and hence $q \circ i = \pi_M \circ \Psi \circ \Psi^{-1} \circ e = \pi_M \circ e = \mathrm{id}_M$.
Conversely, suppose that $q : Z \to M$ is a section of i. Now define $\Psi : Z \to M \oplus N$ by $\Psi(z) = (q(z), p(z))$. Let $(m,n) \in M \oplus N$ be arbitrary, then by surjectivity of p there

exists $z \in Z$ such that $p(z) = n$. As $q \circ i = \mathrm{id}_M$ and $p \circ i = 0$ we then have

$$\Psi(z + i(m - q(z))) = (q(z + i(m - q(z))), p(z + i(m - q(z)))) = (q(z) + m - q(z), n) = (m, n).$$

Hence Ψ is surjective. On the other hand, if we suppose $\Psi(z) = 0$, then in particular $z \in \ker p = \mathrm{im}\, i$, so we can write $z = i(m)$ for some $m \in M$. But then $0 = q(z) = m$, so in fact $m = 0$ and thus $z = 0$. Hence Ψ is an isomorphism. As $\Psi \circ i = e$ and $\pi \circ \Psi = p$, we then obtain that the diagram commutes.

□

Exercise 4. ♠ Let R be a commutative ring. The *projective dimension* of an R-module M is the smallest integer $n \geq 0$ such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0$$

of M. We write $\mathrm{projdim}(M) = n$, and if no finite projective resolution exists, this number is by definition ∞.

In our case, we focus on the ring $R = k[x, y]/(x^2 - y^3)$ and $M = R/(x, y)$. The goal is to show that M does not have finite projective dimension. Proceed as follows:

(1) Compute the dimension as a k-vector space of $\mathrm{Ext}^1_R(M, M)$.
(2) Show that there is a short exact sequence

$$0 \to M \to R/y \to M \to 0.$$

(3) Use the two points above to show that $\mathrm{Ext}^i_R(M, M) \neq 0$ for all $i \geq 0$.
(4) Conclude that $\mathrm{projdim}(M) = \infty$.

Remark 0.1. A celebrated theorem of Serre states that a ring R is *regular* if and only if every module M over R has finite projective dimension. Without going into details, regular means that the associated algebraic variety looks "good" (e.g. would be a smooth manifold over the complex numbers). This gives a very important application of Ext-functors in commutative algebra, since they help detect the projective dimension of modules (and hence regularity of the ring).

In the case above, note that the associated variety (here $\{(x, y) \in \mathbb{R}^2 | x^2 = y^3\}$ if $k = \mathbb{R}$) doesn't look good at the origin (draw this curve!), it has a so-called cusp singularity, and hence it is not regular. This exercise is then about verifying Serre's theorem in a special example.

Proof. (1) To compute $\mathrm{Ext}^1(M, M)$, we will find the first few pieces of a resolution of M. Consider the sequence

$$R^2 \xrightarrow{\psi} R^2 \xrightarrow{\pi} R \longrightarrow M,$$

where $R \to M$ is the quotient map, and we set $\pi(a, b) = ax + by$ and $\psi(a, b) = a(y, -x) + b(x, -y^2)$. It is immediate to see that the kernel of $R \to M$ is $(x, y) = \mathrm{im}(\pi)$, and $\mathrm{im}(\psi) \subseteq \ker(\pi)$. Hence, in order to show that the sequence above is exact, we have to show that $\ker(\pi) = \langle (y, -x), (x, -y^2) \rangle$.

Let us do that now, so consider two polynomials $f, g \in k[x, y]$ such that $xf + gy = (x^2 - y^3)h$ for some $h \in k[x, y]$. Re-ordering, we get that

$$x(f - xh) = -y(g + y^2 h).$$

Since x and y are coprime in the UFD $k[x,y]$, we obtain that x divides $g+y^2h$ and y divides $f-xh$. In other words, we can write

$$\begin{cases} g+y^2h = xp_1; \\ f-xh = yp_2. \end{cases}$$

(and thus automatically $p_2 = -p_1$). We then obtain that

$$(f,g) = h(x,-y^2) + p_2(y,-x),$$

which concludes the proof that $\ker(\pi) = \langle (y,-x),(x,-y^2)\rangle$.

Applying $\mathrm{Hom}(-,M)$ to the piece of the projective resolution of M that we found gives a sequence

$$M^2 \xleftarrow{\psi'} M^2 \xleftarrow{\pi'} M,$$

(recall that $\mathrm{Hom}(R^n,M) \cong M^n$ by sending a map $f\colon R^n \to M$ to $(f(e_1),\dots,f(e_n)) \in M^n$). Furthermore, an explicit computation with the explicit isomorphism we just wrote shows that $\pi'(\overline{1}) = (\overline{x},\overline{y}) = 0$, so $\pi' = 0$. Similarly, we get that $\psi'(1,0) = (\overline{y},\overline{x}) = 0$ and $\psi'(0,1) = (-\overline{x},-\overline{y}^2) = 0$, so again $\psi' = 0$. This shows that $\mathrm{Ext}^1(M,M) \cong M^2$, so its dimension over k is 2.

(2) The quotient map $R \to M$ factors through $R/y \to M$, and its kernel is then $K = (x,y))/(y)$. First, note that $K \neq 0$. Indeed, otherwise we could write

$$x = fy + g(x^2 - y^3)$$

in $k[x,y]$. However, setting $y = 0$ in this equation gives $x = gx^2$ in $k[x]$, which is impossible.

Now, consider the surjection $R \to K$ given by sending 1 to $\overline{x}$. Then its kernel certainly contains y, but also x since its image is $x^2 = y^3 \in (y)$. Thus, the kernel of $R \to K$ contains the maximal ideal (x,y), so since $K \neq 0$, we deduce that $K \cong R/(x,y) = M$.

(3) First of all, let us show that

$$0 \to R \xrightarrow{\cdot y} R \to R/y \to 0$$

is exact. The only thing to show is that y is a non-zero divisor (i.e. the first map is injective), but this follows from the fact that $x^2 - y^3 \in k[x,y]$ is an irreducible polynomial (seen as a element in $k[x][y]$, it has degree 2 but no root).

The associated long exact sequence is Ext-modules shows that:

$$\begin{cases} \mathrm{Hom}(R/y,M) \cong M; \\ \mathrm{Ext}^1(R/y,M) \cong M; \\ \mathrm{Ext}^i(R/y,M) = 0 \text{ for all } i \geq 2. \end{cases}$$

Let us apply $\mathrm{Hom}(-,M)$ to the sequence in (2). Then the long exact in Ext-modules and our computation right above gives that

$$\mathrm{Ext}^i(M,M) \cong \mathrm{Ext}^{i+1}(M,M)$$

for all $i \geq 2$, and a sequence

$$\ldots \longrightarrow \mathrm{Ext}^1(R/y, M) \longrightarrow \mathrm{Ext}^1(M, M) \longrightarrow \mathrm{Ext}^2(M, M) \longrightarrow 0.$$

Since $\mathrm{Ext}^1(R/y, M) \cong M$ has k-dimension 1 and $\mathrm{Ext}^1(M, M)$ has k-dimension 2 by the first point, we obtain that the map $\mathrm{Ext}^1(R/y, M) \to \mathrm{Ext}^1(M, M)$ cannot be surjective, so $\mathrm{Ext}^2(M, M) \neq 0$ by exactness. Thus, we have obtained that $\mathrm{Ext}^i(M, M) \neq 0$ for all $i \geq 0$.

(4) If M had a finite projective resolution, then by definition, we would obtain that for any R-module N, $\mathrm{Ext}^i_R(N, M) = 0$ for $i \gg 0$. This contradicts the previous point.

□

Exercise 5. Consider the ring $\mathbb{Z}[\sqrt{-5}]$.

(1) Is the ideal $(2, 1 + \sqrt{-5})$ a free $\mathbb{Z}[\sqrt{-5}]$-module?
[*Hint:* Consider the element $6 \in \mathbb{Z}[\sqrt{-5}]$.]

(2) Prove that $(2, 1 + \sqrt{-5})$ is a projective $\mathbb{Z}[\sqrt{-5}]$-module.
[*Hint*: Prove that $(2, 1+\sqrt{-5})$ is projective by showing that it is a direct summand of a free module. To do this, define the obvious surjection $p : \mathbb{Z}[\sqrt{-5}]^2 \to (2, 1 + \sqrt{-5})$ and examine the assignment $s : (2, 1+\sqrt{-5}) \to \mathbb{Z}[\sqrt{-5}]^2$ defined by $s(x) = 2xe_1 - \frac{1-\sqrt{-5}}{2}xe_2$.]

Proof. (1) The $\mathbb{Z}[\sqrt{-5}]$-module $I = (2, 1 + \sqrt{-5})$ is not free. Suppose the contrary, then $I \cong \mathbb{Z}[\sqrt{-5}]^{\oplus\Omega}$ for some index set Ω. As I can be generated by 2 elements, we must have $|\Omega| \leq 2$ (to see this, try to prove that a generating set of $R^{\oplus n}$ always contains at least n elements (Hint: you know this for fields, so try to reduce to this case by dividing by a maximal ideal)).
Suppose that $|\Omega| = 2$. Then we have a surjection $\mathbb{Z}[x]^{\oplus 2} \twoheadrightarrow I \cong \mathbb{Z}[x]^{\oplus 2}$ given by mapping $(1, 0)$ to 2 and $(0, 1)$ to $1 + \sqrt{-5}$. But then by Exercise 4 on Sheet 2, this surjection must be an isomorphism, which contradicts the fact that $(3, -1 + \sqrt{-5}) \in \mathbb{Z}[x]^{\oplus 2}$ is mapped to 0.
So we must have $|\Omega| = 1$. We first show that $1 \notin I$ by proving that for all elements $a + b\sqrt{-5} \in I$ we have that $a \equiv b \bmod 2$. We calculate $(r_1 + r_2\sqrt{-5})(1 + \sqrt{-5}) = r_1 - 5r_2 + (r_1 + r_2)\sqrt{-5}$. We have that $r_1 - 5r_2 \equiv r_1 + r_2 \bmod 2$. Obviously $a \equiv b \bmod 2$ for all elements $a + b\sqrt{-5} \in (2)$ hence it is sufficient to note that if $r_1 + r_2\sqrt{-5}$ and $s_1 + s_2\sqrt{-5}$ are such that $r_1 \cong r_2 \bmod 2$ and $s_1 \cong s_2 \bmod 2$ then $(r_1 + r_2\sqrt{-5}) + (s_1 + s_2\sqrt{-5}) = r_1 + s_1 + (r_2 + s_2)\sqrt{-5}$ satisfies $s_1 + r_1 \cong s_2 + r_2 \bmod 2$.
Now suppose that $(a + b\sqrt{-5}) = I$. For any $\alpha = \alpha_1 + \alpha_2\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ write $N(\alpha) = \alpha\bar{\alpha} \in \mathbb{Z}$ where $\bar{\alpha} = \alpha_1 - \alpha_2\sqrt{-5}$. Then N is multiplicative, so $N(a + b\sqrt{-5}) = a^2 + 5b^2$ divides $N(2) = 4$ and $N(1 + \sqrt{-5}) = 6$. This implies $N(a + b\sqrt{-5})$ is either one or two. The equation $a^2 + 5b^2 = 2$ is easily seen to have no integer solutions. If $N(a + b\sqrt{-5}) = 1$ then $1 \in I$ which we have already proven not to be the case, hence the claim follows.

(2) Following the suggestion in the exercise we define $p : \mathbb{Z}[\sqrt{-5}]^2 \to (2, 1 + \sqrt{-5})$ by mapping the canonical basis e_1, e_2 to $e_1 \mapsto 2$ and $e_2 \mapsto 1 + \sqrt{-5}$. If we can prove that p admits a section s we are done by Exercise 3 on this sheet.

Claim: for all $x \in I$ we have that $\frac{1-\sqrt{-5}}{2}x \in \mathbb{Z}[\sqrt{-5}]$.

Proof of claim: write $x = r_1 2 + r_2(1+\sqrt{-5})$, then $\frac{1-\sqrt{-5}}{2}x = (1-\sqrt{-5})r_1 + 3r_2$. Hence the assignment s given in the hint is well-defined. Moreover, we have that $p(s(x)) = p(2xe_1 - \frac{1-\sqrt{-5}}{2}xe_2) = 4x - 3x = x$.

□

Exercise 6. Prove the following.

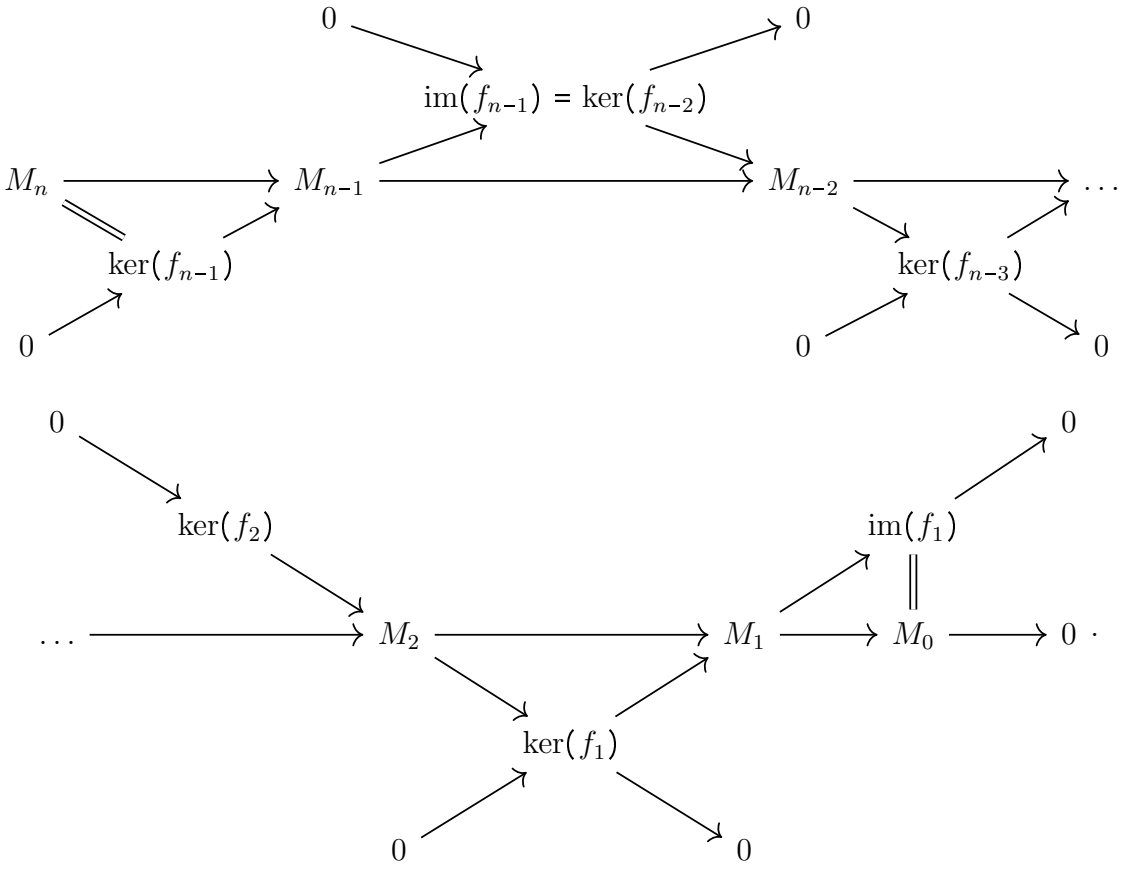
(1) If $0 \longrightarrow M_n \longrightarrow \ldots \longrightarrow M_0 \longrightarrow 0$ is an exact sequence of finitely generated modules over an Artinian and Noetherian ring R, then $0 = \sum_{i=0}^{n}(-1)^i \operatorname{length} M_i$.

(2) Let $R = k[\varepsilon]$ denote (as usual) the quotient $k[x]/(x^2)$ where k is a field (and ε is the class of x). Let M be the R-module $R/(\varepsilon)$. Show that M has no finite resolution by finitely generated free modules.

(3) In general if R is Artinian and Noetherian, and length $R \nmid$ length M, prove that M has no finite resolution by finitely generated free modules.

(4) Prove that over a PID every finitely generated module has a finite free resolution.

Proof. (1) This follows from the additivity of lengths proven in a previous exercise (Exercise 2.4) after slicing the long exact sequence into short exact sequences. Since $\ker(f_i) = \operatorname{im}(f_{i+1})$ for $1 \le i \le n-1$ we get an exact commuting diagram as follow:



By the additivity of lengths on short exact sequences, we have length(M_0) = length(M_1) − length$(\ker(f_1))$ and length$(\ker(f_i))$ = length(M_{i+1}) − length$(\ker(f_{i+1}))$ for $1 \le i \le$

$n-2$. Finally $\text{length}(\ker(f_{n-1}) = \text{length}(M_n)$. These equations combined yield then the formula.

(2) Suppose that

$$0 \longrightarrow R^{\oplus n_k} \xrightarrow{f_k} \dots \xrightarrow{f_2} R^{\oplus n_1} \xrightarrow{f_1} k \longrightarrow 0$$

is a finite length free resolution of k. Then by the previous exercise and by Example 3.2.9 of the lecture notes we have $1 = \sum_{i=1}^{k}(-1)^{i+1}2n_i$, but this is impossible since the right-hand side is an even number.

(3) Suppose that

$$0 \longrightarrow R^{\oplus n_k} \xrightarrow{f_k} \dots \xrightarrow{f_2} R^{\oplus n_1} \xrightarrow{f_1} M \longrightarrow 0$$

is a finite length free resolution of M. Then by the previous exercise we have $\text{length}(M) = \sum_{i=1}^{k}(-1)^{i+1}\,\text{length}(R)n_i$. Since $\text{length}(R)$ divides the right hand side the result follows.

(4) This follows from the structure theorem for finitely generated modules over principal ideal domains. Let $R^{\oplus s} \twoheadrightarrow M$ be a surjection, which exists as M is finitely generated. As R is Noetherian, the kernel K is finitely generated too. But then as R is a domain, K can't have non-trivial torsion elements. From the classification of finitely generated modules, we conclude that $K \cong R^{\oplus t}$ for some t. Hence we obtain an exact sequence

$$0 \to R^{\oplus t} \to R^{\oplus s} \to M \to 0$$

which is thus a finite free resolution of M.

□

Exercise 7. In this exercise R is an integral domain which is not a field; in particular it is commutative. Recall the definition of an R-module M being divisible: for all $m \in M$ and $r \in R \setminus \{0\}$ there exists an $n \in M$ such that $rn = m$. In other words, M is divisible if and only if multiplication by r on M is surjective for every $r \in R \setminus \{0\}$.

(1) Show that a non-trivial free R-module is not divisible.

(2) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module, or in general $\text{Frac}(R)$ is not a projective R-module.
[*Hint:* Define the notion of submodule of divisible elements, and refine (1) by showing that it is trivial for free R-modules.]

(3) From now on, let M, N be R-modules. Let $P_\bullet$ be a projective resolution of M and let $\psi : N \to N$ be the R-module homomorphism corresponding to multiplication by a fixed $r \in R$. Show that ψ induces a co-chain morphism $\text{Hom}_R(P_\bullet, N) \to \text{Hom}_R(P_\bullet, N)$. By passing to cohomology, one obtains a map $\text{Ext}^i_R(M, \psi) : \text{Ext}^i_R(M, N) \to \text{Ext}^i_R(M, N)$. Show that $\text{Ext}^i_R(M, \psi)$ is still just multiplication by r on $\text{Ext}^i_R(M, N)$. In particular, it is independent of the projective resolution.
[*Remark:* One can in fact perform an analogous construction for *any* R-module homomorphism $\psi : N \to L$, and thus obtain a map $\text{Ext}^i_R(M, \psi) : \text{Ext}^i_R(M, N) \to \text{Ext}^i_R(M, L)$, which as in Remark 5.4.26 of the printed course notes is independent of the projective resolution. This makes also $\text{Ext}^i_R(M, -)$ a functor, while in the course we only saw that $\text{Ext}^i_R(-, N)$ is a functor.]

(4) Fix $r \in R$, and let $\phi : M \to M$ be the multiplication by r. Show that $\text{Ext}^i_R(\phi, N)$, as in Definition 5.4.25 of the course notes, is also just the multiplication by r on $\text{Ext}^i_R(M, N)$.

(5) Show that, despite $\mathrm{Frac}(R)$ being not a projective R-module, if N is an R-module such that $\mathrm{Ann}(N) \neq 0$, then $\mathrm{Ext}^i_R(\mathrm{Frac}(R), N) = 0$ for all $i \geq 0$ (note that for P projective, $\mathrm{Ext}^i_R(P, N) = 0$ for all $i > 0$ by definition).

Proof. (1) In view of the hint in the second point, for an R-module M we define

$$\mathrm{Div}(M) := \{m \in M \mid \forall r \in R \setminus \{0\}\ \exists n \in N :\ rn = m\}.$$

One checks easily that this is in fact a submodule of M, and by definition it is clear that M is divisible if and only if $M = \mathrm{Div}(M)$. Now consider a free module $R^{\oplus\Omega}$ where Ω is non-empty. As R is not a field, there exists $r \in R \setminus \{0\}$ which is not a unit. Let $(x_\alpha)_{\alpha\in\Omega} \in \mathrm{Div}(R^{\oplus\Omega})$ and suppose that there is an $\beta \in \Omega$ such that $x_\beta \neq 0$. By definition, we find (y_α) such that $rx_\beta \cdot (y_\alpha) = (x_\alpha)$. In particular we obtain $rx_\beta y_\beta = x_\beta$, which implies that r is a unit, contradiction. Thus $\mathrm{Div}(R^{\oplus\Omega}) = 0$.

(2) We directly prove the general statement. If by contradiction $\mathrm{Frac}(R)$ is projective, then it is a direct summand of a free module F. But then as $\mathrm{Frac}(R)$ is divisible, it injects into $\mathrm{Div}(F)$, which by (1) is trivial. This is a contradiction.

(3) Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & \mathrm{Hom}_R(P_2, N) & \xleftarrow{-\circ p_2} & \mathrm{Hom}_R(P_1, N) & \xleftarrow{-\circ p_1} & \mathrm{Hom}_R(P_0, N) & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle \psi\circ -} & & \downarrow{\scriptstyle \psi\circ -} & & \downarrow{\scriptstyle \psi\circ -} & & \\
\cdots & \longleftarrow & \mathrm{Hom}_R(P_2, N) & \xleftarrow[-\circ p_2]{} & \mathrm{Hom}_R(P_1, N) & \xleftarrow[-\circ p_1]{} & \mathrm{Hom}_R(P_0, N) & \longleftarrow & 0
\end{array}$$

It commutes because post-composition commutes with pre-composition. Notice also that $\psi \circ -$ is just multiplication by r on $\mathrm{Hom}_R(P_i, N)$. Now to get the maps induced on cohomology, we restrict and corestrict to the kernels of the horizontal maps, and then quotient out the images of the horizontal maps. Under all of these operations, multiplication by r remains multiplication by r. Hence the induced map $\mathrm{Ext}^i_R(M, \psi)$ is multiplication by r on $\mathrm{Ext}^i_R(M, N)$.

(4) We follow the construction of $\mathrm{Ext}^i_R(\phi, N)$ as in Definition 5.4.25 of the printed course notes. In a first step, we have to lift the map $\phi : M \to M$ to a chain morphism $\Phi_\bullet : P_\bullet \to P_\bullet$, as in Theorem 5.4.20 of the course notes. Notice that the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r\times} & & \downarrow{\scriptstyle r\times} & & \downarrow{\scriptstyle r\times} & & \\
\cdots & \longrightarrow & P_2 & \xrightarrow[p_2]{} & P_1 & \xrightarrow[p_1]{} & P_0 & \longrightarrow & 0
\end{array}$$

where vertical arrows are multiplication by r, commutes, because multiplication by r commutes with any R-module homomorphism by definition. As in the previous point, this then also induces multiplication by r on homology, so it induces the map $\phi : M \to M$ (recall that M is the 0-th homology module of $P_\bullet$). Therefore, if $\Phi_\bullet$ is multiplication by r on every module of the sequence, then this is a lift of ϕ as in Theorem 5.4.20.

The next step is to apply $\mathrm{Hom}_R(-, N)$ to the entire diagram above. This will reverse all arrows, and the vertical arrows will be pre-composition with multiplication by r. But as again multiplication by r commutes with any R-module homomorphism, the vertical arrows will again be multiplication by r. As in the previous point, the induced

morphism on cohomology is then also just multiplication by r. Hence $\mathrm{Ext}^i_R(\phi, N)$ is multiplication by r on $\mathrm{Ext}^i_R(M, N)$.

(5) Let $r \in \mathrm{Ann}(N) \setminus \{0\}$. Let $\phi : \mathrm{Frac}(R) \to \mathrm{Frac}(R)$ be multiplication by r, then this is an automorphism of $\mathrm{Frac}(R)$. As functors preserve isomorphisms (explained at the end), $\mathrm{Ext}^i_R(\phi, N)$ is still an automorphism, and by the previous point it is multiplication by r on $\mathrm{Ext}^i_R(\mathrm{Frac}(R), N)$.
On the other hand, let $\psi : N \to N$ be multiplication by r. As $r \in \mathrm{Ann}(N)$, this coincides with multiplication by 0. By point (3), we then obtain that multiplication by r on $\mathrm{Ext}^i_R(\mathrm{Frac}(R), N)$ coincides with multiplication by 0 on $\mathrm{Ext}^i_R(\mathrm{Frac}(R), N)$. But above we obtained that multiplication by r is an automorphism. Therefore, we conclude $\mathrm{Ext}^i_R(\mathrm{Frac}(R), N) = 0$ for all $i \geq 0$.
Now we explain what is meant by 'functors preserve isomorphisms'. In fact, one can verify that $\mathrm{Ext}^i_R(\mathrm{id}_M, N) = \mathrm{id}_{\mathrm{Ext}^i_R(M,N)}$ and $\mathrm{Ext}^i_R(\alpha \circ \alpha', N) = \mathrm{Ext}^i_R(\alpha', N) \circ \mathrm{Ext}^i_R(\alpha, N)$ for any M, N, and any R-module homomorphisms $\alpha : M \to M'$ and $\alpha' : M' \to M''$. This is in fact part of the definition of a (contravariant) functor.
Now let $\alpha : M \to M'$ be an isomorphism, with inverse $\alpha' : M' \to M$. Then we have

$$\mathrm{id}_{\mathrm{Ext}^i_R(M,N)} = \mathrm{Ext}^i_R(\alpha' \circ \alpha, N) = \mathrm{Ext}^i_R(\alpha, N) \circ \mathrm{Ext}^i_R(\alpha', N)$$

and

$$\mathrm{id}_{\mathrm{Ext}^i_R(M',N)} = \mathrm{Ext}^i_R(\alpha \circ \alpha', N) = \mathrm{Ext}^i_R(\alpha', N) \circ \mathrm{Ext}^i_R(\alpha, N).$$

Hence $\mathrm{Ext}^i_R(\alpha, N) : \mathrm{Ext}^i_R(M', N) \to \mathrm{Ext}^i_R(M, N)$ is an isomorphism with inverse $\mathrm{Ext}^i_R(\alpha, N)$. So functors preserve isomorphisms.

□