EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 2 - Solutions	

There was one bonus exercise on this problem sheet. The exercise was denoted by the symbol ♠ next to the exercise number.

Exercise 1. Show that the following holds for an R-module M of finite length $l(M)$ (i.e., an R-module M that admits a composition series of finite length).

(1) If there is a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, then $l(M) = l(M') + l(M'')$.

(2) If $N <_R M$ is a proper submodule then $l(N) < l(M)$.

(3) Use (2) to show that any strict chain of submodules in M (not necessary a maximal chain, i.e. not necessarily a composition series) has length smaller than or equal to $l(M)$. Conclude that a module M is of finite length if and only if M is both Noetherian and Artinian.

Proof. (1) The solution has two steps: first we prove that both M' and M'' have finite length, and then we prove the formula.

For the first step, let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$ be a composition series of M (in particular $t = l(M)$). Up to isomorphism (of short exact sequences), we can view M' as an actual submodule of M and $M'' = M \big/ M'$ as the actual quotient of M by M'. Now for $0 \leq i \leq t$, define $M_i' = M' \cap M_i$ and $M_i'' = (M_i + M') \big/ M'$; we would like to understand the quotients of consecutive terms.

On the one hand, we have a natural map $M_{i+1}' \hookrightarrow M_{i+1} \twoheadrightarrow M_{i+1} \big/ M_i$, and the kernel of this composition is exactly M_i'. Hence we obtain an induced inclusion $M_{i+1}' \big/ M_i' \hookrightarrow M_{i+1} \big/ M_i$. As the latter is simple, we obtain that $M_{i+1}' \big/ M_i'$ is either trivial or simple.

On the other hand, we have by the third isomorphism theorem that $M_{i+1}'' \big/ M_i'' \cong (M_{i+1} + M) \big/ (M_i + M)$. Then, we have a natural map
$M_{i+1} \hookrightarrow M_{i+1} + M \twoheadrightarrow (M_{i+1} + M) \big/ (M_i + M)$, and the composed arrow is easily seen to be surjective. Also, M_i is included in the kernel of the composition, so we obtain an induced surjective map $M_{i+1} \big/ M_i \twoheadrightarrow (M_{i+1} + M) \big/ (M_i + M) \cong M_{i+1}'' \big/ M_i''$. As $M_{i+1} \big/ M_i$ is simple, we obtain that $M_{i+1}'' \big/ M_i''$ is either trivial or simple.

In conclusion, the quotients of consecutive terms both in $M_0' \subseteq \cdots \subseteq M_t'$ and $M_0'' \subseteq \cdots \subseteq M_t''$ are all either simple or trivial. So by deleting some of the modules in the sequence, we will obtain composition series both for M' and M''. Hence M' and M'' have finite length (and length smaller than or equal to t).

Now for the second step, by the one-to-one correspondence of submodules of M'' and submodules of M containing M' it is clear that a composition series for M' can be extended to a composition series for M by adding the preimage of a composition series of M''. This gives a composition series for M of length $l(M') + l(M'')$. Therefore, since by the Jordan Holder Theorem $l(M)$ is the length of *any* composition series, we obtain $l(M') + l(M'') = l(M)$.

(2) Follows directly from the argument above.

(3) Let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ be a strict chain of length n. Then by (2) we have $l(M) > l(M_{n-1}) > \cdots > l(M_0) = 0$, hence $l(M) \geq n$. Since every chain of M is of finite length bounded by $l(M)$, M is both Noetherian and Artinian. The implication in the other direction was discussed in Remark 3.2.4 of the lecture notes.

□

Exercise 2. Let R be a ring and let M be a finitely generated module over R. Let $f : M \to M$ be an R-module homomorphism.

(1) Suppose that R is a Noetherian ring.
   (i) Does injectivity of f implies surjectivity?
   (ii) Does surjectivity of f implies injectivity?
   (iii) What happens if R is not necessarily Noetherian?
   Hint: For one of the directions, try to reduce to the Noetherian case by considering the $\mathbb{Z}$-subalgebra of R generated by finitely many suitable elements.

(2) Suppose that M is a module of finite length, show that f is injective if and only if f is surjective.

Proof. (1) (i) Let R be a ring with $a \in R$ neither a unit nor a zero divisor, then multiplication by a is an injective but not surjective morphism $m_a : R \to R$.

(ii) Suppose that M is a finitely generated module over a Noetherian ring, then M is Noetherian. Let $f : M \to M$ be a surjective morphism. For all k we have containments $\ker(f^k) \subset \ker(f^{k+1})$. Therefore, there exists a positive integer m such that $\ker(f^{m+1}) = \ker(f^m)$. In particular, $f : \operatorname{im}(f^m) \to M$ is injective, but by surjectivity $\operatorname{im}(f^m) = M$, therefore f is injective.

Remark 0.1. Amazingly, the statement remains true even if R is not Noetherian. Let us prove it now. Let e_i for $1 \leq i \leq n$ be generators of M as an R-module. Let $f(e_i) = \sum_{i=1}^n a_{ij} e_j$ for all i. By surjectivity there exists b_{jk} such that $e_j = \sum_{k=1}^n b_{jk} f(e_k)$ for all j. Suppose that $m \in \ker(f)$ with $m = \sum_i m_i e_i$. Let $\mathbb{Z}[a_{ij}, b_{ij}, m_k] \to R$ be the natural inclusion morphism, where $\mathbb{Z}[a_{ij}, b_{ij}, m_k]$ is the $\mathbb{Z}$-subalgebra of R generated by the a_{ij}'s, b_{ij}'s and m_k's. There is therefore an induced structure of $R' = \mathbb{Z}[a_{ij}, b_{ij}, m_k]$-module on M. Let M' be the R'-submodule generated by e_i for $1 \leq i \leq n$. By definition of M' the morphism f induces a morphism $f' : M' \to M'$, it is surjective since $e_i = f(\sum_k b_{ik} e_k)$. As now R' is Noetherian (it is a finitely generated $\mathbb{Z}$-algebra), we obtain by the previous point that the element $m \in \ker(f')$ is zero. As $m \in \ker(f)$ was arbitrary, we conclude that f is injective.

Later, we will see a very important statement called *Nakayama's lemma*. This will provide a very easy proof of this fact over any (commutative) ring, without relying on Noetherian approximation as above.

(2) Consider the short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0 .$$

By Exercise 1.1, we have $l(M) = l(\ker(f)) + l(\mathrm{im}(f))$. Since the zero module is the only module of length zero, f being surjective implies that $\ker(f) = 0$. Converserly, if f is injective, then $l(M) = l(\mathrm{im}(f))$, hence $l(\mathrm{im}(f))$ can not be a proper submodule of M by the same exercise, i.e. $M = \mathrm{im}(f)$.

□

Exercise 3. (1) Let R be a PID, and let $f \in R$ be a product of $n \geq 0$ prime elements. Prove that the length of $R/(f)$ as an R-module is equal to n.

(2) Let $f \in \mathbb{R}[x]$ be a nonzero polynomial with exactly $n \geq 0$ non-real roots (counted with multiplicity). Prove that

$$\dim_{\mathbb{R}}\left(\mathbb{R}[x]/(f)\right) - \mathrm{length}_{\mathbb{R}[x]}\left(\mathbb{R}[x]/(f)\right) = n/2$$

(3) Let M be a $\mathbb{Z}$-module. Prove that M has finite length if and only if it is finite (as a set).

(4) Give an example of a ring and a module over this ring which has finite length but infinitely many submodules.

Proof. (1) We prove the assertion by induction on n; for $n = 0$ it clearly holds, and for $n = 1$ it holds since primes in a PID are maximal, and thus the quotient of R by a prime is simple.

So assume that we have shown the assertion for some $n \geq 1$, and let f be a product of $n + 1$ primes. Let p be a prime dividing f and write $f = pg$ where g is a product of n primes. Then we have a natural surjection $R/(f) \twoheadrightarrow R/(g)$ of R-modules, and let K be the kernel. It is straightforward to see that $K = R\cdot(g + (f))$. Now we have a short exact sequence

$$0 \to \mathrm{Ann}_R(g + (f)) \to R \xrightarrow{\cdot(g+(f))} K \to 0.$$

Finally, one can easily verify that $\mathrm{Ann}_R(g + (f)) = (p)$, and thus $K \cong R/(p)$. As we then have a short exact sequence

$$0 \to R/(p) \to R/(f) \to R/(g) \to 0,$$

it follows from Exercise 1.1 and the induction hypothesis that $R/(f)$ has length $n + 1$.

(2) The dimension of $\mathbb{R}[x]/(f)$ as an $\mathbb{R}$-vector space is $d = \deg f$. Furhtermore, as $\mathbb{R} \subseteq \mathbb{C}$ is a field extension of degree 2, the irreducible polynomials of $\mathbb{R}[x]$ are the linear polynomials and the quadratic polynomials having no real roots. Therefore, if m is the number of real roots of f counted with multiplicity, one can see that f is the product of exactly $m + n/2$ irreducible polynomials. Hence by the previous exercise we obtain that the length of $\mathbb{R}[x]/(f)$ is equal to $m + n/2$. As $d = m + n$, we obtain

$$\dim_{\mathbb{R}}\left(\mathbb{R}[x]/(f)\right) - \mathrm{length}_{\mathbb{R}[x]}\left(\mathbb{R}[x]/(f)\right) = m + n - (m + n/2) = n/2.$$

(3) If M is finite as a set then M has finite length as there are only finitely many submodules. Conversely, if M has finite length, then by Exercise 1 it is in particular Noetherian, so finitely generated. By the classification of finitely generated $\mathbb{Z}$-modules, we have an isomorphism $M \cong \mathbb{Z}^{\oplus r} \oplus F$ for some finite $\mathbb{Z}$-module F and $r \geq 0$. If by contradiction $r \geq 1$, then M contains a copy of $\mathbb{Z}$ as a submodule, so again by Exercise 1 we obtain that $\mathbb{Z}$ has finite length. This is not true, e.g. as $\mathbb{Z}$ is not Artinian. Hence $r = 0$ and $M \cong F$ is finite.

(4) It suffices to take an infinite field k and a finite dimensional k-vector space V of dimension greater than or equal to 2. It is clearly of finite length, and if $v_1, v_2 \in V$ are linearly independent, then $\{k \cdot (v_1 + \lambda v_2)\}_{\lambda \in k}$ is an infinite family of distinct subspaces. $\square$

Exercise 4. ∘ Let $n, m > 0$ be integers, let k be a field and let $R := k[x, y]$. Show that the R-module

$$M := k[x,y] \big/ (x^n, y^m)$$

has length nm.

Hint: Exercise 1 can be useful to decompose this computation into easier ones, allowing some induction argument. The same applies for the next point.

∘ Let $p > 0$ be a prime number. Compute the length of

$$\mathbb{Z}[x] \big/ (p^2, x^2 - p),$$

as a module over the ring $\mathbb{Z}[x]$.

Proof. (1) First let us show the following: for any $d \geq 0$, the module

$$N_d := k[x,y] \big/ (x, y^d)$$

has length d. Set

$$S := k[x,y] \big/ (x)$$

and $\pi : R \to S$ the quotient map. By Exercise 2.3 on sheet 1, we can define an S-module structure on N_d such that for all $r \in R$ and $n \in N_d$, $r \cdot n = \pi(r) \cdot n$.

With this in mind, it is immediate that S-submodules of N_d are the same as R-submodules of N_d, so in particular its length is unchanged.

Now, $S \cong k[y]$ by setting $x = 0$, and through this isomorphism we see that N_d corresponds to

$$k[y] \big/ (y^d)$$

so we know by Exercise 3.1 that its length is d.

Now, let us compute the length of

$$N_{n,m} := k[x,y] \big/ (x^n, y^m)$$

is nm. If $n = 1$, this was already worked out before, so assume $n \geq 2$. Consider the morphism $\phi : k[x, y] \to N_{n,m}$ given by sending 1 to $x^{n-1} + (x^n, y^m)$. Note that the sequence

$$k[x,y] \xrightarrow{\phi} N_{n,m} \to N_{n-1,m} \to 0$$

is exact where $N_{n,m} \to N_{n-1,m}$ is the usual quotient map, so we obtain a short exact sequence

$$0 \to k[x,y] \big/ \ker(\phi) \to N_{n,m} \to N_{n-1,m} \to 0$$

Let us understand $\ker(\phi)$. Clearly, $(x, y^m) \subseteq \ker(\phi)$, and given $a \in \ker(\phi)$, we get that by definition there exists $b, c \in k[x, y]$ such that

$$x^{n-1} a = x^n b + y^m c$$

In particular x^{n-1} divides $y^m c$, so since x and y are coprime ($k[x,y]$ is a UFD) we get that x^{n-1} divides c (write $c = x^{n-1}c'$). Thus,

$$a = xb + y^m c'$$

or in other words $a \in (x, y^m)$.

Hence we have proven that $\ker(\phi) = (x, y^m)$, so we finally have a short exact sequence

$$0 \to N_{1,m} \to N_{n,m} \to N_{n-1,m} \to 0$$

which by induction on n gives us

$$l(N_{n,m}) = l(N_{n-1,m}) + l(N_{1,m}) = (n-1)m + m = nm.$$

(2) Let $M := \mathbb{Z}[x]/(p^2, x^2 - p)$, and consider the quotient map

$$\pi\colon \mathbb{Z}[x]\big/(p^2, x^2 - p) \to \mathbb{Z}[x]\big/(p, x^2 - p).$$

Note that the latter module is isomorphic to

$$N := (\mathbb{Z}/p\mathbb{Z}[x])\big/(x^2),$$

and since the $\mathbb{Z}[x]$-action this module factors through an action of $\mathbb{Z}/p\mathbb{Z}[x]$, let us compute the length of N as a $\mathbb{Z}/p\mathbb{Z}[x]$-module. Since this ring is a PID, we deduce by Exercise 3.(1) that the length of N is 2.

Let us compute $\ker(\pi)$. By the third isomorphism theorem,

$$\ker(\pi) = (p, x^2 - p)\big/(p^2, (x^2 - p)),$$

so in particular it is generated by $\overline{p}$ (i.e. the class of p in the quotient). Hence, we have a surjection

$$\theta\colon \mathbb{Z}[x] \to \ker(\pi),$$

sending 1 to $\overline{p}$.

Let us understand $\ker(\theta)$. It is immediate to see that $(p, x^2 - p) \subseteq \ker(\theta)$. On the other hand, if $f(x) \in \ker(\theta)$, then $pf(x) = p^2 a(x) + (x^2 - p)b(x)$ for some $a(x), b(x) \in \mathbb{Z}[x]$. In particular, p divides $(x^2 - p)b(x)$, so since p is prime, p divides $b(x)$. Write $b(x) = pb'(x)$. Then

$$f(x) = pa(x) + (x^2 - p)b'(x) \in (p, x^2 - b).$$

Thus, we have proven that

$$(p, x^2 - p) = \ker(\theta).$$

In other words,

$$\ker(\pi) \cong \mathbb{Z}[x]\big/(p, x^2 - p) \cong N.$$

Equivalently, we have a short exact sequence

$$0 \to N \to M \to N \to 0$$

so by additivity of the length,

$$\text{length}(M) = 2\,\text{length}(N) = 4.$$

$\square$

Exercise 5. ♠ Compute the length of the $\mathbb{C}[x,y,z]$-module module

$$M := \mathbb{C}[x,y,z]\Big/\left(x^3+3x^2+2xy, y^2-1, z^{2024}\right).$$

Exercise 6. Let R be a Noetherian ring. Are the following rings Noetherian? Are they Artinian?

(1) $R[x,\frac{1}{x}] := \{\sum_{i=-m}^{n} a_i x^i : a_i \in R, m,n \in \mathbb{N}\}$.
(2) $R[x_1,x_2,x_3,...]$.
(3) $R[[x]]$, the ring of formal power series[1] with coefficients in R.
Hint: For an ideal I and each $n \in \mathbb{N}$, let $I_n := \{a_n : \exists \sum_{i=n}^{\infty} a_i x^i \in I\}$. Then adapt the proof of the Hilbert basis theorem.
(4) $C^0(\mathbb{R})$, the ring of continuous functions $\mathbb{R} \to \mathbb{R}$ with pointwise operations.
(5) $\mathbb{R}[x]\Big/\left((x-1)^2x\right)$.

Proof. (1) We will show that $R[x,\frac{1}{x}]$ is isomorphic to a quotient of a polynomial ring. It then follows that it is Noetherian by the Hilbert basis theorem (as Noetherianity is preserved under quotients).
The isomorphism in question comes from the R-algebra homomorphism

$$\phi : R[u,v] \to R[x,\frac{1}{x}]$$
$$u \mapsto x,\ v \mapsto \frac{1}{x},$$

which exists by the universal property of $R[u,v]$. This is surjective as any element of $R[x,1/x]$ can be written as some polynomial in x and $\frac{1}{x}$ by definition. Thus it has some kernel I, and hence $R[x,\frac{1}{x}] \cong R[u,v]/I$ is Noetherian.

As a side note, we can go further, and identify the kernel $\ker\phi = I$ to be the ideal $(uv-1)$. For it is clear that $uv-1 \in I$, and suppose that $g \in \ker\phi$. Then we can use elements of $(uv-1)$ to cancel mixed terms, and so write $g = g_1 + g_2$ where $g_1 \in (uv-1)$ and $g_2 = \sum_{i\geq 0} a_i u^i + \sum_{j>0} b_j v^j$ for some $a_i, b_j \in R$. But it is clear that g_2 cannot be in $\ker\phi$ unless all of its coefficients are zero. So $g = g_1 \in (uv-1)$.

Take $R \neq 0$ to be any Noetherian ring. There is an infinite descending chain of ideals in $R[x,x^{-1}]$ given by $(x+1) \supsetneq ((x+1)^2) \supseteq ((x+1)^3) \supseteq \ldots$. We need to prove that the containment is strict. To this end suppose that there exists a $k > 0$ such that $((x+1)^k) = ((x+1)^{k+1})$. Then there exists $f \in k[x,x^{-1}]$ such that $(x+1)^k = f(x,x^{-1})(x+1)^{k+1}$. Write $f(x,x^{-1}) = \sum_{m\leq i\leq n} a_i x^i$ with $m \leq n$ integers and $a_m, a_n \neq 0$. Then there is a term of degree $k+n+1$ with coefficient $a_n \neq 0$ on the right-hand side, and thus $m \leq n < 0$ as the left-hand side has only terms of degree less than or equal to k. But then there is a non-zero term of degree $m < 0$ on the right-hand side corresponding to $a_m x^m$. This is not possible, since the left-hand side has no non-zero term with negative degree. We conclude that $f = 0$, but this amounts to a contradiction since $(x+1)^k \neq 0$ since it has non-zero coefficients corresponding to the terms x^k and 1. Hence $R[x,\frac{1}{x}]$ isn't Artinian.

[1] $R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$, where multiplication and addition are defined formally, as what you think they should be. These are purely formal objects: there is no requirement for any kind of convergence.

(2) $R[x_1, x_2, ...]$ is not Noetherian, as the ideal $(x_1, x_2, ...)$ cannot be finitely generated. It is not Artinian (for any choice of $R \neq 0$), since it contains the strictly descending chain $(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \dots$.

(3) $R[[x]]$ is not Artinian (for any choice of $R \neq 0$), since it contains the strictly descending chain $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \dots$.

$R[[x]]$ is Noetherian, and the proof is a variant of the proof of the Hilbert basis theorem.

To this end suppose I is an ideal of $R[[x]]$. For each integer $n \geq 0$, let

$$I_n := \{a_n : \exists \sum_{i=n}^{\infty} a_i x^i \in I\}.$$

For each n, this is an ideal of R, and by multiplying each power series by x we see that $I_n \subseteq I_{n+1}$ for each n. So by the ascending chain condition, there is M such that $I_n = I_{n+1}$ for all $n \geq M$.

Also, for each $i \leq M$, I_i is finitely generated, so we may fix a finite set $\{a_{i,j}\}_{0 \leq j \leq N}$ of generators for I_i (we take always the same number N of generators by repeating elements if needed). For each $0 \leq i \leq M$ and $0 \leq j \leq N$, fix $f_{i,j} \in I$ such that

$$f_{i,j} = a_{i,j} x^i + \text{ higher order},$$

which exsits by construction of I_i.

We claim that the the ideal J generated by the set $\{f_{i,j}\}_{\substack{0 \leq i \leq M \\ 0 \leq j \leq N}}$ is equal to I. Let $g = \sum_{k=0}^{\infty} b_k x^k \in I$. By construction of I_0, we can find an element $h_0 \in J$ having the same term of order 0 as g: there exists an R-linear combination of $a_{0,0}, \dots, a_{0,N}$ equal to b_0, and taking h_0 to be the same R-linear combination of $f_{0,0}, \dots, f_{0,N}$ will do. Similarly, we can find an element $h_1 \in J$ having the same term of order 1 as $g - h_0$. Iterating this procedure, we construct an element $h = h_0 + \dots + h_{M-1} \in J$ such that $g - h$ has no terms of degree strictly smaller than M.

Now we proceed similarly, but with a slight modification. As before, we can find coefficients $c_{0,0}, \dots, c_{0,N} \in R$ such that $l_0 = c_{0,0} f_{M,0} + \dots + c_{0,N} f_{M,N}$ has the same term of order M as $g - h$. Then, we can find $c_{1,0}, \dots, c_{1,N} \in R$ such that $l_1 = c_{1,0} x f_{M,0} + \dots + c_{1,N} x f_{M,N}$ (we added a factor x in there to make things of the correct order; in the next step we will need a factor x^2 and so on) has same term of order $M + 1$ as $g - h - l_0$. We iterate this procedure indefinitely, and for $0 \leq j \leq N$ define the power series $c_j = \sum_{k \geq 0} c_{k,j} x^k$, as well as $l = c_0 f_{M,0} + \dots + c_N f_{M,N} \in J$. One can then show by comparing coefficients that $g - h - l = 0$. As $h, l \in J$, we conclude $g \in J$, and as $g \in I$ was arbitrary, we obtain $I = J$. Hence I is finitely generated, and thus $R[[x]]$ is Noetherian.

(4) $C^0(\mathbb{R})$ is neither Artinian nor Noetherian. To this end define $I_n = \{f \in C(\mathbb{R}) : f(x) = 0 \text{ for all } x \geq n\}$, where $n \in \mathbb{Z}$. It is clear that $I_n \subset I_{n+1}$. We need to show that the containment is strict. To this end, define for example the continuous function f by $f(x) = 0$ for all $x \geq n+1$ and $f(x) = x - (n+1)$ for all $x \leq n+1$, this is a well-defined continuous function $f \in I_{n+1} \setminus I_n$. So $(I_n)_{n \in \mathbb{Z}}$ is a strictly increasing sequence of ideals indexed by $\mathbb{Z}$, showing that $C^0(\mathbb{R})$ is neither Artinian nor Noetherian.

(5) The most efficient solution is the following: it suffices to notice that the dimension of $\mathbb{R}[x]\big/\big((x-1)^2 x\big)$ as an $\mathbb{R}$-vector space is equal to 3 (the degree of the polynomial), so

in particular it is finite. As ideals of $\mathbb{R}[x]\big/\big((x-1)^2x\big)$ are in particular $\mathbb{R}$-subspaces, and finite dimensional vector spaces obviously satisfy the ascending and descending chain conditions, we obtain that $\mathbb{R}[x]\big/\big((x-1)^2x\big)$ is both Artinian an Noetherian.

$\square$