EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 11 - Solutions	

There was one bonus exercise on this problem sheet. The exercise was denoted by the symbol ♠ next to the exercise number.

Exercise 1. Let R be a ring and let M, N be R-modules. Prove that $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$. [*Hint:* Try to adapt the proof of Proposition 5.3.8 in the printed course notes.]

Proof. Let $P_\bullet \to M$ be a projective resolution. Notice that we have a short exact sequence $0 \to \mathrm{im}(p_1) \xrightarrow{i} P_0 \xrightarrow{p_0} M \to 0$. By right exactness of the tensor product, it follows that $\mathrm{im}(p_1) \otimes_R N \to P_0 \otimes_R N \to M \otimes_R N \to 0$ is exact. To conclude, it suffices to verify that the image of $i \otimes_R \mathrm{id}_N$ coincides with the image of $p_1 \otimes_R \mathrm{id}_N$. For this, notice that $p_1 = i \circ p_1|^{\mathrm{im}(p_1)}$, where $p_1|^{\mathrm{im}(p_1)}$ is the corestriction of p_1 to $\mathrm{im}(p_1)$. Hence

$$p_1 \otimes_R \mathrm{id}_N = (i \otimes_R \mathrm{id}_N) \circ (p_1|^{\mathrm{im}(p_1)} \otimes_R \mathrm{id}_N),$$

but $p_1|^{\mathrm{im}(p_1)}$ is surjective and thus by right exactness also $p_1|^{\mathrm{im}(p_1)} \otimes_R \mathrm{id}_N$, and thus $\mathrm{im}(p_1 \otimes_R \mathrm{id}_N) = \mathrm{im}(i \otimes_R \mathrm{id}_N)$. Hence we have

$$M \otimes_R N \cong {P_0 \otimes_R N}/{\mathrm{im}(i \otimes_R \mathrm{id}_N)} = {P_0 \otimes_R N}/{\mathrm{im}(p_1 \otimes_R \mathrm{id}_N)} = H_0(P_\bullet \otimes_R N) = \mathrm{Tor}_0^R(M,N).$$

□

Exercise 2. Let R be a ring and N an R-module. We say that N is *flat* if for every short exact sequence of R-modules

$$0 \to M \to M' \to M'' \to 0$$

the sequence

$$0 \to M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to 0$$

is exact. Prove that the following are equivalent:

(1) N is flat,
(2) $\mathrm{Tor}_i^R(M,N) = 0$ for every R-module M and every $i > 0$,
(3) $\mathrm{Tor}_1^R(M,N) = 0$ for every R-module M.

[*Hint:* For (1)⇒(2) take a free resolution of M and tensor it with N to compute the Tor-functors. For (3)⇒(1) use the long exact sequence for left derived functors.]

Proof. We prove a cycle of implications:

(1) ⇒ (2) : Let $P_\bullet \to M$ be a projective resolution of some R-module M. As N is flat, the chain complex $(P_\bullet \to M) \otimes_R N$ (with the M at position -1) is still exact, and thus its homology groups vanish. Thus for $i > 0$ we obtain

$$\mathrm{Tor}_i^R(M,N) = H_i(P_\bullet) = H_i(P_\bullet \to M) = 0.$$

(2) ⇒ (3) : Trivial.

(3) ⇒ (1) : Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of R-modules. From the long exact sequence for left derived functors, we obtain an exact sequence

$$\cdots \to \mathrm{Tor}_1^R(M',N) \to \underbrace{\mathrm{Tor}_1^R(M'',N)}_{=0} \to M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to 0.$$

In particular, $0 \to M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to 0$ is exact, and thus N is flat. $\square$

Exercise 3. Let $R = k[x,y]$ where k is a field. Consider the R-modules $M := (x,y)$ (i.e. the ideal generated by x and y) and $N := R/M$.

(1) Compute $\operatorname{Tor}_i^R(M,N)$ for all integers $i \geq 0$.
[*Hint:* Use the definition.]
(2) Is N flat?
(3) Compute $\operatorname{Tor}_i^R(N,N)$ for all integers $i \geq 0$.
[*Hint:* Use the long exact sequence.]

Proof. (1) We saw already a couple of times that M admits the free resolution $P_\bullet \to M$ given by

$$\begin{array}{ccccccc} 0 \longrightarrow & P_1 = R & \longrightarrow & R \oplus R = P_0 & \longrightarrow & M & \longrightarrow 0 \\ & 1 & \longmapsto & (y, -x) & & & \\ & & & (1,0) & \longmapsto & x & \\ & & & (0,1) & \longmapsto & y. & \end{array}$$

This already shows that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 2$. Furthermore, we have $\operatorname{Tor}_1^R(M,N) = \ker(p_1 \otimes_R \mathrm{id}_N)$. Notice that $p_1 \otimes_R \mathrm{id}_N$ maps a simple tensor $r \otimes n$ to $(ry, -rx) \otimes n$, and

$$(ry, -rx) \otimes n = (ry, 0) \otimes n - (0, rx) \otimes n = (r, 0) \otimes (\underbrace{yn}_{=0}) - (0, r) \otimes (\underbrace{xn}_{=0}) = 0.$$

Hence $p_1 \otimes_R \mathrm{id}_N$ is equal to 0 on simple tensors, and thus equal to 0. We therefore obtain $\operatorname{Tor}_1^R(M,N) \cong R \otimes_R N \cong N$. Also, as then $\operatorname{im}(p_1 \otimes_R \mathrm{id}_N) = 0$ we have $\operatorname{Tor}_0^R(M,N) \cong (R \oplus R) \otimes_R N \cong N \oplus N$. In conclusion

$$\operatorname{Tor}_i^R(M,N) \cong \begin{cases} N \oplus N & \text{if } i = 0, \\ N & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) We have $\operatorname{Tor}_1^R(M,N) = N \neq 0$ and thus N isn't flat by Exercise 2.
(3) Notice that we have a short exact sequence $0 \to M \to R \to N \to 0$. We would like to tensor this with N and take the induced long exact sequence. To prepare this, notice that $\operatorname{Tor}_i^R(R,N) = 0$ for all integers $i > 0$. Indeed, a projective resolution of R is provided by $\cdots \to 0 \to R \xrightarrow{\mathrm{id}} R \to 0$. As we have the 0 module on positions with index $i > 0$, and this remains the case after tensoring with N, we conclude that indeed $\operatorname{Tor}_i^R(R,N) = 0$ for all integers $i > 0$. For $i > 1$, consider now the following excerpt from the long exact sequence

$$\cdots \to \underbrace{\operatorname{Tor}_i^R(R,N)}_{=0} \to \operatorname{Tor}_i^R(N,N) \to \operatorname{Tor}_{i-1}^R(M,N) \to \underbrace{\operatorname{Tor}_{i-1}^R(R,N)}_{=0} \to \cdots.$$

Hence we obtain that $\operatorname{Tor}_i^R(N,N) \cong \operatorname{Tor}_{i-1}^R(M,N)$ for all integers $i > 1$, and thus by point (1) we have $\operatorname{Tor}_i^R(N,N) = 0$ for all integers $i > 2$ and $\operatorname{Tor}_2^R(N,N) \cong N$. Now we focus on the start of the long exact sequence:

$$(*)\qquad \cdots \to \underbrace{\mathrm{Tor}_1^R(R,N)}_{=0} \to \mathrm{Tor}_1^R(N,N) \to M \otimes_R N \to R \otimes_R N \to N \otimes_R N \to 0.$$

The key observation here is that the map $M \otimes_R N \to R \otimes_R N$ is the zero map. Indeed, if $r \otimes (s + M) \in M \otimes_R N$ is a simple tensor then this is mapped to

$$r \otimes (s + M) = 1 \otimes (\underbrace{r(s + M)}_{=0}) = 0$$

insinde $R \otimes_R N$. So as simple tensors generate the tensor product, we indeed have that the map $M \otimes_R N \to R \otimes_R N$ is trivial. Plugging this back into $(*)$ we directly obtain $\mathrm{Tor}_1^R(N,N) \cong M \otimes_R N \cong N \oplus N$, where we used the previous point and Exercise 6. Finally, as the image of $M \otimes_R N$ inside $R \otimes_R N$ is 0, we obtain also from $(*)$ that $R \otimes_R N \to N \otimes_R N$ is an isomorphism. Hence $\mathrm{Tor}_0^R(N,N) \cong N \otimes_R N \cong N$. In conclusion,

$$\mathrm{Tor}_i^R(N,N) \cong \begin{cases} N & \text{if } i \in \{0,2\}, \\ N \oplus N & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Exercise 4. Let R be a ring.

(1) Prove that free R-modules are flat.
(2) Prove that projective R-modules are flat.
[*Hint:* Use the characterization of projective modules as direct summands of free modules.]
(3) Assume that R is an integral domain. Determine for which ideals I of R the R-module R/I is flat.

Remark 0.1. There exists a partial converse of (2): a flat finitely generated module over a Noetherian ring is projective.

The finite generation hypothesis is very important, as the $\mathbb{Z}$-module $\mathbb{Q}$ is flat (see exercise 6.3), but not projective. There are also counter-examples in the Non-noetherian case.

Proof. (1) It suffices to prove that $R^{\oplus I}$ is flat, where I is an arbitrary set. Notice that for an R-module M, we have a natural isomorphism $\eta_M : M \otimes_R R^{\oplus I} \to M^{\oplus I}$, given on simple tensors by $m \otimes (r_i)_i \mapsto (r_i m)_i$. Indeed, η_M exists as it is the map induced by the R-bliniear map $(m, (r_i)_i) \in M \oplus R^{\oplus I} \mapsto (r_i m)_i \in M^{\oplus I}$. We now construct an inverse: let $\theta_M : M^{\oplus I} \to M \otimes_R R^{\oplus I}$ be the map defined by sending $(m_i)_i$ to $\sum_{j : m_j \neq 0} m_j \otimes (\delta_{ij})_i$. It is straightforward to verify that this is the inverse of η_M. Lastly, note that η_M is natural. To see this, let $f : M \to N$ is an R-module homomorphism. We must verify that $f^{\oplus I} \circ \eta_M = \eta_N \circ (f \otimes_R \mathrm{id}_{R^{\oplus I}})$. It suffices to verify this on simple tensors: the LHS maps $m \otimes (r_i)_i$ via $(r_i m)_i$ to $(f(r_i m))_i$, and the RHS maps $m \otimes (r_i)_i$ via $f(m) \otimes (r_i)_i$ to $(r_i f(m))_i$. These two agree as as f is R-linear.

Now to show that $R^{\oplus I}$ is flat, it suffices to show that $- \otimes_R R^{\oplus I}$ preserves injections (as we already know that it is right exact by Exercise 5 of sheet 10). So let $f : M \to N$ be injective, then by what we showed above, under the identifications $M \otimes_R R^{\oplus I} \cong M^{\oplus I}$

and $N \otimes_R R^{\oplus I} \cong N^{\oplus I}$, the map $f \otimes_R \mathrm{id}_{R^{\oplus I}}$ is just $f^{\oplus I}$. So as $f^{\oplus I}$ is injective, $f \otimes_R \mathrm{id}_{R^{\oplus I}}$ is too, and hence $R^{\oplus I}$ is flat.

(2) Suppose M is projective and let M' be an R-module such that $M \oplus M' \cong R^I$. In a similar way as for the previous point, if A is an R-module, then there is a natural isomorphism $\eta_A : A \otimes_R (M \oplus M') \to (A \otimes_R M) \oplus (A \otimes_R M')$ which maps $a \otimes (m, m')$ to $(a \otimes m, a \otimes m')$. Under this identification, if $f : A \to B$ is an R-linear map, then $f \otimes_R \mathrm{id}_{M \oplus M'}$ corresponds to $(f \otimes_R \mathrm{id}_M) \oplus (f \otimes_R \mathrm{id}_{M'})$; it suffices to check this on simple tensors.
Now if f is injective, then by the previous point $f \otimes_R \mathrm{id}_{M \oplus M'}$, and thus under the identifications provided by $\eta_\bullet$ the map $(f \otimes_R \mathrm{id}_M) \oplus (f \otimes_R \mathrm{id}_{M'})$ is injective. In particular, $f \otimes_R \mathrm{id}_M$ is injective. Hence $- \otimes_R M$ preserves injections, which proves that M is flat.

(3) If $I = 0$ then $R/I = R$ is flat. If $I = R$ then $R/I = 0$ is also flat. We will show that $R/I = 0$ is flat only in these two cases. Let $I \subset R$ be a non-zero proper ideal and let $a \in I$ be non-zero. Since R is a domain the R-module morphism $m_a : R \to R$ defined by $m_a(r) = ar$ is injective. However, if we apply $- \otimes_R R/I$ and identify $R \otimes_R R/I \cong R/I$, we obtain $m_a \otimes_R \mathrm{id}_{R/I} : R/I \to R/I$ which maps $r + I$ to $ar + I = 0$. Therefore $m_a \otimes \mathrm{id}_{R/I}$ is the zero map, which is not injective since $I \neq R$, hence R/I is not flat.

□

Exercise 5. Let R be a ring containing a multiplicatively closed subset T, and let M be an R-module. Show that there is an isomorphism of R-modules

$$T^{-1}M \cong T^{-1}R \otimes_R M.$$

Further show that this is an isomorphism of $T^{-1}R$-modules.
[*Remark:* The right hand side naturally has the structure of a $T^{-1}R$-module by point (1) of Exercise 6 on Sheet 10.]

Proof. Let $\psi : T^{-1}R \otimes_R M \to T^{-1}M$ be defined as being induced from the bilinear map $T^{-1}R \oplus M \to T^{-1}M$ given by $(\frac{r}{t}, m) \mapsto \frac{rm}{t}$; that the latter is well-defined and bilinear is direct. In formulas, ψ is given on simple tensors by $\frac{r}{t} \otimes m \mapsto \frac{rm}{t}$
Defining an inverse to ψ can be done by hand (by mapping m/t to $(1/t) \otimes m$ and showing that it is well-defined and a morphism), and this approach will be given first. A more conceptual approach is to prove a universal property for $T^{-1}M$, similar to the one in Theorem 9.2.3 of the notes, that allows to construct a map out of $T^{-1}M$ from a map out of M. This is stated in Remark 9.2.8, and proven below the approach by hand.

First the approach by hand. We show that $g : T^{-1}M \to T^{-1}R \otimes_R M$ defined by $g(\frac{m}{t}) = \frac{1}{t} \otimes m$ for $m \in M$ and $t \in T$ is well-defined and inverse to ψ. Suppose that $\frac{m_1}{t_1} = \frac{m_2}{t_2}$. Then there is $t' \in T$ such that $t'(t_2 m_1 - t_1 m_2) = 0$. Thus $\frac{1}{t_1} \otimes m_1 = \frac{t' t_2}{t' t_2 t_1} \otimes m_1 = \frac{1}{t' t_2 t_1} \otimes t' t_2 m_1 = \frac{1}{t' t_2 t_1} \otimes t' t_1 m_2$, which is equal to $\frac{1}{t_2} \otimes m_2$ by a symmetrical argument. This shows that g is well defined. To show that it is a $T^{-1}R$-module homomorphism, we must show that it respects addition and scalar multiplication: for addition, $g(\frac{m_1}{t_1} + \frac{m_2}{t_2}) = g(\frac{t_2 m_1 + t_1 m_2}{t_1 t_2}) = \frac{1}{t_1 t_2} \otimes (t_2 m_1 + t_1 m_2) = \frac{1}{t_1 t_2} \otimes t_2 m_1 + \frac{1}{t_1 t_2} \otimes t_1 m_2 = \frac{1}{t_1} \otimes m_1 + \frac{1}{t_2} \otimes m_2$ as required. For scalar multiplication, $g(\frac{r}{s}\frac{m}{t}) = \frac{1}{st} \otimes rm = \frac{r}{st} \otimes m = \frac{r}{s}(\frac{1}{t} \otimes m) = \frac{r}{s}\phi(\frac{m}{t})$. Now it remains to show that ψ and g are mutually inverse: we have

$$\psi(g(\frac{m}{t})) = \psi(\frac{1}{t} \otimes m) = \frac{m}{t}$$

and on simple tensors (it suffices to check these as they generate the tensor product)

$$g(\psi(\frac{r}{t} \otimes m)) = g(\frac{rm}{t}) = \frac{1}{t} \otimes rm = \frac{r}{t} \otimes m.$$

So g and ψ are isomorphisms, and as g is $T^{-1}R$-linear, ψ is too. As a side note, notice that it follows from this isomorphism that every element of $T^{-1}R \otimes_R M$ is expressible as a simple tensor.

Conceptual approach:

Theorem. *Let R be a ring with multiplicatively closed subset T and let M be an R-module. Let $i : M \to T^{-1}M$ be the R-module homomorphism defined by $m \mapsto \frac{m}{1}$. Lastly, an R-module N will be called T-invertible if for every $t \in T$ the multiplication map $\mu_t : n \in N \mapsto tn \in N$ is an isomorphism.*

(1) A T-invertible R-module N admits a natural $T^{-1}R$-module structure, defined by $\frac{r}{t} \cdot n := \mu_t^{-1}(rn)$.

(2) For every R-module homomorphism $\phi : M \to N$ with N being T-invertible, there exists a unique R-module homomorphism $\overline{\phi} : T^{-1}M \to N$ such that $\phi = \overline{\phi} \circ i$. Furthermore, $\overline{\phi}$ is a $T^{-1}R$-module homomorphism for the $T^{-1}R$-module structure on N from the previous point.

Proof. (1) The R-module structure is equivalent to a ring homomorphism $\lambda : R \to \mathrm{End}_{\mathbb{Z}}(N)$, mapping r to μ_r. The ring $\mathrm{End}_{\mathbb{Z}}(N)$ isn't necessarily commutative, so to get around this let $S \subseteq \mathrm{End}_{\mathbb{Z}}(N)$ be the subring generated by $\lambda(R)$ and $\{\mu_t^{-1} \mid t \in T\}$. Then it is straightforward to check that S is commutative, and that the corestriction $\lambda|^S : R \to S$ maps every element of T to a unit. Hence, by the universal property of $T^{-1}R$, there exists a map $\Lambda : T^{-1}R \to S \hookrightarrow \mathrm{End}_{\mathbb{Z}}(N)$ extending $\lambda|^S$. This gives N the structure of a $T^{-1}R$-module, and it is straightforward to check that $\frac{r}{t} \cdot n := \mu_t^{-1}(rn)$.

(2) We define $\overline{\phi} : T^{-1}M \to N$ by the formula $\overline{\phi}\left(\frac{m}{t}\right) = \frac{1}{t}\phi(m)$ (where we make use of the $T^{-1}R$-module structure on N). We have to check that this is well defined: suppose $\frac{m_1}{t_1} = \frac{m_2}{t_2}$, i.e. there is a $t' \in T$ such that $t'(t_2m_1 - t_1m_2) = 0$. Then by applying ϕ we obtain $t'(t_2\phi(m_1) - t_1\phi(m_2)) = 0$ inside N, and as N is T-invertible this implies $\frac{1}{t_1}\phi(m_1) = \frac{1}{t_2}\phi(m_2)$. Hence $\overline{\phi}$ is well-defined. Note that $\phi = \overline{\phi} \circ i$ follows immediately from the construction. So what is left to check is that $\overline{\phi}$ is a $T^{-1}R$-module homomorphism:

(additive): $\overline{\phi}\left(\frac{m_1}{t_1} + \frac{m_2}{t_2}\right) = \frac{1}{t_1t_2}\phi(t_2m_1 + t_1m_2) = \frac{1}{t_1}\phi(m_1) + \frac{1}{t_2}\phi(m_2) = \overline{\phi}\left(\frac{m_1}{t_1}\right) + \overline{\phi}\left(\frac{m_2}{t_2}\right)$ for all $t_1, t_2 \in T$ and $m_1, m_2 \in M$.

($T^{-1}R$-linear): $\overline{\phi}\left(\frac{r}{s}\frac{m}{t}\right) = \frac{1}{st}\phi(rm) = \frac{r}{st}\phi(m) = \frac{r}{s}\overline{\phi}\left(\frac{m}{t}\right)$ for all $r \in R$, $s, t \in T$ and $m \in M$.

Hence $\overline{\phi}$ is a $T^{-1}R$-module homomorphism (and in particular an R-module homomorphism).

□

With this at hand, notice that $T^{-1}R \otimes_R M$ is T-invertible: indeed, by Exercise 6.1 on Sheet 10, $T^{-1}R \otimes_R M$ has the structure of a $T^{-1}R$-module (such that multiplication by $r/1$ is multiplication by r). In particular, multiplication by $t \in T$ is invertible (the inverse being multiplication by $\frac{1}{t}$). Therefore, by the universal property of $T^{-1}M$, the map $\phi : M \to T^{-1}R \otimes_R M$ which sends $m \mapsto 1 \otimes m$ induces $\overline{\phi} : T^{-1}M \to T^{-1}R \otimes_R M$ defined by $\frac{m}{t} \mapsto \frac{1}{t}(1 \otimes m) = \frac{1}{t} \otimes m$. It is then easy to see that $\overline{\phi}$ is inverse to ψ, and as $\overline{\phi}$ is a $T^{-1}R$-module homomorphism, ψ is too. □

Exercise 6. Let R be a ring with multiplicative subset T, and suppose that L, M and N are R-modules.

(1) Show that if there is an R-module homomorphism $f : M \to N$ then there is a natural $T^{-1}R$-module homomorphism $f_T : T^{-1}M \to T^{-1}N$.
(2) Show that there is an isomorphism of $T^{-1}R$-modules $T^{-1}(M \oplus N) \cong (T^{-1}M) \oplus (T^{-1}N)$.
(3) Suppose there is an exact sequence
$$0 \to L \to M \to N \to 0.$$
Prove that the sequence
$$0 \to T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$$
is also exact. Deduce that if $L \subset M$ is a sub R-module, then $T^{-1}\left(M/L\right) \cong T^{-1}M \big/ T^{-1}L$ and that localization by T is an exact functor of R-modules and that $T^{-1}R$ is a flat R-module.
(4) Let $\mathfrak{p}$ be a prime ideal of R. Show that there is an isomorphism of rings $\mathrm{Frac}\left(R/\mathfrak{p}\right) \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.
[*Remark:* For a local ring A with maximal ideal $\mathfrak{m}$ we call $A/\mathfrak{m}$ the residue field of A.]

Proof. There are two possible approaches to the first three points: either one uses the universal property of localisation of a module proven in the conceptual solution to Exercise 5, or one uses the description of localisation of a module by a tensor product provided by Exercise 5. Both have their advantages and disadvantages, so will discuss both.

(1) *Tensor approach:* By applying the functor $T^{-1}R \otimes_R -$ we obtain a map $\mathrm{id}_{T^{-1}R} \otimes_R f : T^{-1}R \otimes_R M \to T^{-1}R \otimes_R N$ which on simple tensors is defined by $\frac{r}{t} \otimes m \mapsto \frac{r}{t} \otimes f(m)$. Under the identification provided by Exercise 5, this gives a map of R-modules $f_T : T^{-1}M \to T^{-1}N$ defined by $\frac{m}{t} \mapsto \frac{f(m)}{t}$. It is then straightforward to check to see that this is a $T^{-1}R$-module homomorphism.
Pure localisation approach: Denote by i_M resp. i_N the natural maps $i_M : M \to T^{-1}M$ and $i_N : N \to T^{-1}N$. Then as $T^{-1}N$ seen as an R-module is T-invertible, the map $i_N \circ f : M \to T^{-1}N$ induces a $T^{-1}R$-module homomorphism $f_T : T^{-1}M \to T^{-1}N$ such that $f_T \circ i_M = i_N \circ f$ (by the universal property of module localisation proven in the solution to Exercise 5). It is straightforward to check that f_T maps $\frac{m}{t} \in T^{-1}M$ to $\frac{f(m)}{t} \in T^{-1}N$.
(2) *Tensor approach:* The functor $L \otimes_R -$ is additive for any R-module L, meaning more precisely that the map $L \otimes_R (M \oplus N) \to (L \otimes_R M) \oplus (L \otimes_R N)$ sending a simple tensor $l \otimes (m, n)$ to $(l \otimes m, l \otimes n)$ is a well-defined isomorphism of R-modules. By applying this to $L = T^{-1}R$ and using the identification provided by Exercise 5, we

obtain that the map $T^{-1}(M \oplus N) \to (T^{-1}M) \oplus (T^{-1}N)$ defined by sending $\frac{(m,n)}{t}$ to $\left(\frac{m}{t}, \frac{n}{t}\right)$ is an R-module isomorphism. It is then straightforward to check that this is in fact a $T^{-1}R$-module homomorphism.

Pure localisation approach: Denote by i_M, i_N resp. $i_{M\oplus N}$ the natural localisation maps. The map $i_M \oplus i_N : M \oplus N \to T^{-1}M \oplus T^{-1}N$ goes to a T-invertible module, and hence by the universal property induces a map of $T^{-1}R$-modules $\phi : T^{-1}(M \oplus N) \to T^{-1}M \oplus T^{-1}N$ such that $\phi \circ i_{M\oplus N} = i_M \oplus i_N$ (which in particular implies that $\frac{(m,n)}{t}$ is mapped to $\left(\frac{m}{t}, \frac{n}{t}\right)$). Now either one checks by hand that this is bijective (which is straightforward), or one constructs an inverse (which is a bit heavy on notation but a good exercise). We will do the latter. If j_M resp. j_N are the natural inclusions $j_M : M \hookrightarrow M \oplus N$ resp. $j_N : N \hookrightarrow M \oplus N$, then the maps $i_{M\oplus N} \circ j_M$ and $i_{M\oplus N} \circ j_N$ induce $T^{-1}R$-maps $\psi_M : T^{-1}M \to T^{-1}(M \oplus N)$ and $\psi_N : T^{-1}N \to T^{-1}(M \oplus N)$ such that $\psi_M \circ i_M = i_{M\oplus N} \circ j_M$ and $\psi_N \circ i_N = i_{M\oplus N} \circ j_N$ (which in particular implies that $\frac{m}{t}$ is mapped to $\frac{(m,0)}{t}$ and $\frac{n}{t}$ is mapped to $\frac{(0,n)}{t}$). Then ψ_M and ψ_N together induce $\psi : T^{-1}M \oplus T^{-1}N \to T^{-1}(M \oplus N)$, given by mapping $\left(\frac{m}{t}, \frac{n}{t'}\right)$ to $\frac{(m,0)}{t} + \frac{(0,n)}{t'}$, which can also be written as $\frac{(t'm,tn)}{tt'}$. It is then straightforward to check that ϕ and ψ are mutually inverse.

(3) We first prove exactness of the sequence.

Tensor approach: As $T^{-1}R \otimes_R -$ is right exact by Exercise 5 on sheet 10, we already have that $T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$ is exact. Let f be the map $f : L \to M$; to conclude, we must show that f_T is injective. So suppose that $\frac{l}{t}$ is mapped to 0 by f_T, i.e. $\frac{f(l)}{t}$ is 0 inside $T^{-1}M$. This means that there is $t' \in T$ such that $t'f(l) = 0$ in M, which by injectivity of f means that $t'l = 0$. But then $\frac{l}{t} = 0$ inside $T^{-1}L$, so f_T is injective. Hence $0 \to T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$.

Pure localisation approach: Denote by $f : L \to M$ and $g : M \to N$ the maps of the sequence. Just as in the tensor approach, one proves that f_T is injective. To show that g_T is surjective, let $\frac{n}{t} \in T^{-1}N$ be arbitrary. Then as g is surjective, there is $m \in M$ such that $g(m) = n$, and thus g_T maps $\frac{m}{t}$ to $\frac{n}{t}$, so g_T is surjective. So it remains to show exactness at $T^{-1}M$. As $g_T \circ f_T$ is equal to $(g \circ f)_T$ which is 0, we obtain $\operatorname{im} f_T \subseteq \ker g_T$. To prove the reverse inclusion, let take $\frac{m}{t} \in \ker g_T$. That is, we have that $\frac{g(m)}{n}$ is 0 inside $T^{-1}N$, i.e. there exists $t' \in T$ such that $t'g(m) = 0$. By exactness of the original sequence, there exists $l \in L$ such that $f(l) = t'm$. Hence we obtain that f_T maps $\frac{l}{tt'}$ to $\frac{m}{t}$. Thus we proved that also $\ker g_T \subseteq \operatorname{im} f_T$, and thus $0 \to T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$ is exact.

Note that for any R-submodule $L \subseteq M$ we can set $N := M/L$ to obtain an exact sequence $0 \to L \to M \to N \to 0$, and then as $0 \to T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$ is also exact we obtain

$$T^{-1}\left(M/L\right) = T^{-1}N \cong T^{-1}M \big/ T^{-1}L.$$

Note that under this isomorphism, $\frac{m+L}{t}$ is mapped to $\frac{m}{t} + T^{-1}L$.

To prove that localisation by T is a (covariant) functor, we must show that $(\mathrm{id}_M)_T =$

$\mathrm{id}_{T^{-1}M}$ and $(g \circ f)_T = g_T \circ f_T$ for any R-module homomorphisms $f : L \to M$ and $g : M \to N$, which are both straightforward. The above then implies that localisation by T is moreover exact.
Finally, the identification provided by Exercise 5 shows that $T^{-1}R \otimes_R -$ is an exact functor, which means that $T^{-1}R$ is a flat R-module.

(4) We construct mutually inverse morphisms. First, notice that the composition $R \to R_\mathfrak{p} \to {R_\mathfrak{p}}/{\mathfrak{p}R_\mathfrak{p}}$ has kernel equal to $\mathfrak{p}$. Indeed, every element of $\mathfrak{p}$ is mapped to 0, and if $r \in R$ is mapped to 0 then $\frac{r}{1}$ is inside $\mathfrak{p}R_\mathfrak{p}$, which means that there exists $r' \in \mathfrak{p}$ and $t \in R \setminus \mathfrak{p}$ such that $\frac{r}{1} = \frac{r'}{t}$. This in turn means that there is $t' \in R \setminus \mathfrak{p}$ such that $t'(rt - r') = 0$. In particular, $rtt' \in \mathfrak{p}$, and as $tt' \notin \mathfrak{p}$ we obtain $r \in \mathfrak{p}$. Therefore, we obtain an injective ring morphism ${R}/{\mathfrak{p}} \to {R_\mathfrak{p}}/{\mathfrak{p}R_\mathfrak{p}}$. Notice that if $t + \mathfrak{p} \neq 0$, then this is mapped to $\frac{t}{1} + \mathfrak{p}R_\mathfrak{p}$. This has inverse $\frac{1}{t} + \mathfrak{p}R_\mathfrak{p}$, so every non-zero element is mapped to an invertible element in ${R_\mathfrak{p}}/{\mathfrak{p}R_\mathfrak{p}}$. Thus the universal property of localisation induces a ring morphism $\mathrm{Frac}\left({R}/{\mathfrak{p}}\right) \to {R_\mathfrak{p}}/{\mathfrak{p}R_\mathfrak{p}}$, mapping $\frac{r+\mathfrak{p}}{t+\mathfrak{p}}$ to $\frac{r}{t} + \mathfrak{p}R_\mathfrak{p}$.
On the other hand, the composition $R \to {R}/{\mathfrak{p}} \to \mathrm{Frac}\left({R}/{\mathfrak{p}}\right)$ maps every element of $R \setminus \mathfrak{p}$ to an invertible element, and hence induces a ring map $R_\mathfrak{p} \to \mathrm{Frac}\left({R}/{\mathfrak{p}}\right)$ given by sending $\frac{r}{t}$ to $\frac{r+\mathfrak{p}}{t+\mathfrak{p}}$. Then, if $r \in \mathfrak{p}$, then $\frac{r}{1}$ is mapped to 0, and thus the ideal generated by elements of this form, i.e. $\mathfrak{p}R_\mathfrak{p}$, is in the kernel. Hence we obtain ${R_\mathfrak{p}}/{\mathfrak{p}R_\mathfrak{p}} \to \mathrm{Frac}\left({R}/{\mathfrak{p}}\right)$ given by sending ${r}/{t} + \mathfrak{p}R_\mathfrak{p}$ to $\frac{r+\mathfrak{p}}{t+\mathfrak{p}}$. This is clearly inverse to the morphism constructed in the previous paragraph, so it is an isomorphism of rings.

□

Exercise 7. Let R be a ring, let S be a multiplicatively closed subset, and let M and N be R-modules. Show that for all $i \geq 0$,

$$S^{-1}\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N).$$

If furthermore R is Noetherian and M is finitely generated, then also

$$S^{-1}\operatorname{Ext}_R^i(M, N) \cong \operatorname{Ext}_{S^{-1}R}^i(S^{-1}M, S^{-1}N).$$

Proof. Let us first show the statement about Tor's. Let $P_\bullet \to M$ be a projective resolution. Note that each $S^{-1}P_i$ is also projective about $S^{-1}R$ (for example use Exercise 5, and the analogous fact for tensor products). Furthermore, by exactness of the functor S^{-1} (see Exercise 6), we deduce that $S^{-1}P_\bullet$ is a projective resolution (over $S^{-1}R$) of $S^{-1}M$.

Before, concluding, let us show that for any R-modules A, B, we have $S^{-1}A \otimes_{S^{-1}R} S^{-1}B \cong S^{-1}(A \otimes_R B)$.

This follows from the computation

$$S^{-1}A \otimes_{S^{-1}R} S^{-1}B \cong A \otimes_R S^{-1}R \otimes_{S^{-1}R} S^{-1}B \cong A \otimes_R S^{-1}B \cong A \otimes_R B \otimes_R S^{-1}R \cong S^{-1}(A \otimes_R B).$$

Combining all this, we deduce that $S^{-1}P_\bullet \otimes_{S^{-1}R} S^{-1}N \cong S^{-1}(P_\bullet \otimes_R N)$. Taking i'th homology (and again using exactness of S^{-1}) shows the statement.

Now, let us show the statement about Ext-functors. The exact same argument will work, once we know that $S^{-1}\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$. First of all, there is always

a natural map
$$\theta_{M,N}\colon S^{-1}\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$
given by sending an element $\frac{f}{s}$ (with $f\colon M \to N$ and $s \in S$) to the map
$$\frac{m}{s'} \longmapsto \frac{f(m)}{ss'}.$$

If $M \cong R^{\oplus m}$ (let $e_1, \dots, e_m$ denote a basis of M), then this map is an isomorphism. Indeed, we have
$$S^{-1}\operatorname{Hom}_R(M,N) \cong S^{-1}\operatorname{Hom}_R(R^{\oplus m}, N) \cong S^{-1}(N^{\oplus m}) \cong (S^{-1}N)^{\oplus m},$$
where the isomorphism sends
$$\frac{f}{s} \longmapsto \left(\frac{f(e_1)}{s}, \dots, \frac{f(e_m)}{s}\right).$$
On the other hand, we also have an isomorphism
$$\operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \cong \operatorname{Hom}_{S^{-1}R}((S^{-1}R)^{\oplus m}, S^{-1}N) \cong (S^{-1}N)^{\oplus m}$$
sending
$$g \longmapsto \left(g\left(\frac{e_1}{1}\right), \dots, g\left(\frac{e_m}{1}\right)\right).$$
We then immediately see that the triangle
$$\begin{tikzcd}
S^{-1}\operatorname{Hom}_R(M,N) \arrow[rr, "\theta_{M,N}"] \arrow[dr, "\cong"'] & & \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \arrow[dl, "\cong"] \\
& (S^{-1}N)^{\oplus m} &
\end{tikzcd}$$
commutes, so $\theta_{M,N}$ is an isomorphism in this case.

For the general case, consider an exact sequence
$$R^{\oplus m_2} \to R^{\oplus m_1} \to M \to 0$$
(recall that M is finitely generated and R is Noetherian). We can then apply $\operatorname{Hom}_R(-,N)$ and then S^{-1} to obtain an exact sequence
$$0 \longrightarrow S^{-1}\operatorname{Hom}_R(M,N) \longrightarrow S^{-1}\operatorname{Hom}_R(R^{\oplus m_1},N) \longrightarrow S^{-1}\operatorname{Hom}_R(R^{\oplus m_2},N).$$
We could also have applied first S^{-1} and then $\operatorname{Hom}_{S^{-1}R}(-, S^{-1}N)$ to obtain
$$0 \longrightarrow \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \longrightarrow \operatorname{Hom}_{S^{-1}R}(S^{-1}R^{\oplus m_1}, S^{-1}N) \longrightarrow \operatorname{Hom}_{S^{-1}R}(S^{-1}R^{\oplus m_2}, S^{-1}N).$$

Our natural maps θ give the following commutative diagram with exact rows:
$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\operatorname{Hom}_R(M,N) \arrow[r] \arrow[d, "\theta_{M,N}"] & S^{-1}\operatorname{Hom}_R(R^{\oplus m_1},N) \arrow[r] \arrow[d, "\theta_{R^{\oplus m_1},N}"] & S^{-1}\operatorname{Hom}_R(R^{\oplus m_2},N) \arrow[d, "\theta_{R^{\oplus m_2},N}"] \\
0 \arrow[r] & \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \arrow[r] & \operatorname{Hom}_{S^{-1}R}(S^{-1}R^{\oplus m_1}, S^{-1}N) \arrow[r] & \operatorname{Hom}_{S^{-1}R}(S^{-1}R^{\oplus m_2}, S^{-1}N)
\end{tikzcd}$$
Since both $\theta_{R^{\oplus m_1},N}$ and $\theta_{R^{\oplus m_2},N}$ are isomorphisms, we deduce by the 5-lemma (Lemma 5.6.2 in the notes) that $\theta_{M,N}$ is an isomorphism. $\square$

Given a commutative ring R, a prime ideal $\mathfrak{p}$ of R is said to be *minimal* it if contains no strict prime ideal. For example, if R is a domain, then (0) is the only minimal prime.

For the following exercise, you may use without proof the following result, which we will see later (although its proof does not require any sophisticated tool):

Proposition 0.2. *Let R be a Noetherian ring. Then R contains at most finitely many minimal prime ideals.*

Exercise 8. ♠ As we have seen in Exercise 3.2 of sheet 1, an Artinian ring has dimension zero. The goal of this exercise is to show the converse. Let R be a zero-dimensional Noetherian ring. Proceed as follows:

(1) Assume that R is local with maximal ideal $\mathfrak{m}$. Show that $\mathfrak{m}^n = 0$ for some $n \geq 1$.
(2) With the same assumptions, conclude that R is Artinian.
(3) Use the local case to show that R is Artinian in general. *Hint: Use the proposition above*
(4) Find an example of a zero-dimension ring which is not Artinian.

Proof. (1) You have seen in "Anneaux et Corps" that $\mathrm{nil}(R)$ is the intersection of all prime ideals of R. Since R has dimension zero and is local, there is a unique prime ideal, namely $\mathfrak{m}$. Hence, all elements of $\mathfrak{m}$ are nilpotent. Since $\mathfrak{m}$ is finitely generated (this is important!), then for some $n \gg 0$, $\mathfrak{m}^n = 0$.

(2) Notice that for all $j > 0$, $\mathfrak{m}^j$ is finitely generated, so $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is a finite-dimensional $R/\mathfrak{m}$-vector space. In particular, it is of finite length. Hence, we have a chain of inclusions

$$0 = \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m} \subseteq R,$$

where all successive quotients have finite length. Since having finite length is stable under extensions, we deduce that also R has finite length, and hence R is Artinian (see Exercise 1 of sheet 2).

(3) Note that for any ideal I and prime ideal $p \subseteq R$, the inclusion $I \subseteq R$ gives an inclusion of localizations $I_p \subseteq R_p$, and it is clear from the definition that $I_p = I^e$. Furthermore, note that R has finitely many prime ideals. Indeed, it has finitely many minimal ones by Noetherianity, but since R has dimension zero, being a minimal prime and being a prime is the same.

Now, to the exercise. Let

$$\cdots \subseteq I_{j+1} \subseteq I_j \subseteq \cdots \subseteq I_0$$

be a descending chain of ideals of R.

Since all localizations at primes R_p are Artinian by the local case and there are only finitely many prime ideals in R, we deduce that for $j \gg 0$, $(I_{j+1})_p = (I_j)_p$ for all primes ideals $p \subseteq R$. By exactness of localizations, this is the same as saying for $(I_j/I_{j+1})_p = 0$ for all p and $j \gg 0$. We then conclude by Exercise 2 of sheet 12 that $I_j/I_{j+1} = 0$ for $j \gg 0$, so in other words this descending chain of ideals stabilizes.

(4) Let k be a field, and set

$$R = k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots).$$

Since $R/\sqrt{0} \cong k$, we obtain that R has a unique prime ideal (i.e. $(x_1, x_2, \dots)$), so surely R has dimension zero.

However, set $I_j = (x_1^2, \dots, x_{j-1}^2, x_j, x_{j+1}, \dots) \subseteq R$. Then these ideals form an infinite strictly descending sequence of ideals, so R is not Artinian.

□