Lecture 13

Before starting this lecture, I would like to give some geometric intuition behind the localization at prime ideals. Let K again be an algebraically closed field, and let $p \subset K[x_1, \cdots, x_n]$ be a prime ideal. Consider $V(p) \subset K^n$ the irreducible algebraic variety associated with p.

Recall that we can interpret elements of $F = \mathrm{Frac}(K[x_1, \cdots, x_n])$ as meromorphic (= rational) functions on K^n. That is, for $f \in F$ there is a Zariski open subset $U \subset K^n$ such that f is well defined over U and yields a morphism $f \colon U \to K$. In fact, using, for example, that $K[x_1, \cdots, x_n]$ is a UFD, one can easily verify that there is a maximal Zariski open U over which f is defined: this is called the domain of definition of f.

Since $K[x_1, \cdots, x_n]$ is a domain, we can see all of its localizations as a subring of F in a natural way. Then, one checks that the localization $K[x_1, \cdots, x_n]_p$ consists precisely of those rational funtions $f \in F$ whose domain of definition U satisfies $U \cap V(p) \neq$. This means that the restriction of f to $V(p)$ is well defined on a Zariski open of $V(p)$: in particular, it can still have poles at some points of $V(p)$. Now, the ideal $p^e \subset K[x_1, \cdots, x_n]_p$ is maximal, as we know from the previous lecture, and this corresponds to all the functions that are constantly zero on $V(p)$. The fact that the ideal is maximal means that $K[x_1, \cdots, x_n]_p/p^e$ is a field, which is precisely the fraction field of $K[x_1, \cdots, x_n]/p$ (i.e., the field of rational functions on $V(p)$).

1 Ideals, integral extensions and localization

In this section we shall prove many propositions regarding the interplay between integral extensions, localizations, and behavior of ideals.

Proposition 1. *Let $R \subset S$ be an integral ring extension, $J \subset S$ an ideal. Then $R/J^c \to S/J$ is injective and integral.*

Proposition 2. *The injectivity is a completely general phenomena: for any ring morphism $R \to S$ and any ideal $J \subset S$, the induced map $R/J^c \to S/J$ is injective.*

The integrality is also easy to prove: pick any $[s] \in S/J$. Since s is integral over R it satisfies a monic equation $s^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 = 0$ for $r_i \in R$. But then looking at this equation in S/J yields $[s]^n + [r_{n-1}][s]^{n-1} + \cdots + [r_1][s] + [r_0] = 0$, which proves the statement.

Proposition 3. *Let $R \subset S$ be an integral ring extension, and let $q \subset S$ be a prime ideal. Then q is maximal if and only if $q \cap R = q^c$ is maximal.*

Proof. We know that q^c is prime. Hence, $R/q^c \subset S/q$ is an extension of domains. But we have proved that R/q^c is a field if and only if S/q is a field. $\square$

Proposition 4. *Let $R \subset S$ be an integral ring extension. Let $T \subset R$ be a multiplicatively closed subset. Then $T^{-1}R \to T^{-1}S$ is injective, and is integral as well.*

Proof. To show that $T^{-1}R \to T^{-1}S$ is injective, assume that $r/t \in T^{-1}R$ is mapped to 0. This means that there is $u \in T$ such that $ru = 0$ in S. Since $R \subset S$ is injective and $T \subset R$, this means that $ru = 0$ in R too, hence that $r/t = 0$.

To prove that the extension is integral, pick $s/t \in T^{-1}S$. Then s is integral over R and we have an equation

$$s^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 = 0$$

divide this by t^n and obtain

$$(s/t)^n + (r_{n-1}/t)(s/t)^{n-1} + \cdots + (r_1/t^{n-1})(s/t) + (r_0/t^n) = 0$$

which shows that s/t is integral over $T^{-1}R$. $\square$

We now prove one of the most fundamental results in this area of commutative algebra:

Theorem 5 (Going-up)**.** *Let $R \subset S$ be integral ring extension.*

1. *For any prime ideal $p \subset R$ there is a prime ideal $q \subset S$ such that $q^c = q \cap R = p$.*
2. *For every $p_1 \subsetneq p_2 \subset R$ prime ideals and every $q_1 \subset S$ prime ideal such that $q_1 \cap R = p_1$, we can choose a prime ideal q_2 such that $q_1 \subset q_2$ and $q_2 \cap R = p_2$.*
3. *If $q_1 \subsetneq q_2 \subset S$ are prime ideals, then $q_1 \cap R \neq q_2 \cap R$.*

Proof. We make use of Theorem 6 of the previous lecture.

- We denote by R_p and S_p the localization of both rings at $R \setminus p$. So we get an integral extension $R_p \subset S_p$ by one of the previous propositions. Now, R_p is a local ring, with pR_p as maximal ideal.

  Pick $m \subset S_p$ any maximal ideal. Then $m \cap R_p$ is maximal as well, because $R_p \subset S_p$ is integral; hence, $m \cap R_p = pR_p$ necessarily. Consider now the following commutative diagram, where the 'iota' maps denote the localizations:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\iota_R"'] & S \arrow[d, "\iota_S"'] \\ R_p \arrow[r, hook] & S_p \end{tikzcd}$$

  due to commutativity, we have

$$p = \iota_R^{-1}(pR_p) = \iota_R^{-1}(R_p \cap m) = R \cap \iota_S^{-1}(m).$$

  Hence, we can put $q = \iota_S^{-1}(m)$.

- Let us localize both R and S at $T = R \setminus p_2$. We get an integral extension $R_{p_2} \subset S_{p_2}$. Now, since $p_1 \subset p_2$ we have $T \cap p_1 = \emptyset$, hence $p_1 R_{p_2}$ is a proper prime ideal of R_{p_2}. Similarly, $q_1 \cap T = q_1 \cap R \cap T = p_1 \cap T = \emptyset$, hence $q_1 S_{p_2}$ is a proper prime ideal of S_{p_1}. Now we simply pick a maximal ideal $q_1 S_{p_2} \subset m \subset S_{p_2}$ and proceed as before.

- Assume that this is false, so that $p = q_1 \cap R = q_2 \cap R$. Again, localize both R and S at $T = R \setminus p$. Now, since $q_1 \cap T = q_2 \cap T = \emptyset$ we have that $q_1 S_p \subsetneq q_2 S_p$ by point 4 of Theorem 6 of the previous lecture. Look again at the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\iota_R"'] & S \arrow[d, "\iota_S"'] \\ R_p \arrow[r, hook] & S_p \end{tikzcd}$$

We have
$$p = q_1 \cap R = \iota_S^{-1}(q_1 S_p) \cap R = \iota_R^{-1}(q_1 S_p \cap R_p)$$
again from point 4 of Theorem 6 of the previous lecture, we deduce that $pR_p = q_1 S_p \cap R_p$. The same holds for q_2, i.e., $pR_p = q_2 S_p \cap R_p$. But pR_p is maximal, and since $S_p \subset R_p$ is integral, this implies that both $q_2 S_p$ and $q_1 S_p$ are maximal. Hence $q_1 S_p = q_2 S_p$, which is a contradiction.

□

Corollary 6. *Let $R \subset S$ be an integral ring extension. Then $\dim(R) = \dim(S)$.*

Proof. Recall that $\dim(R) = \sup\{n \colon \exists p_0 \subsetneq p_1 \subsetneq \cdots p_n \subset R \text{ chain of prime ideals}\}$.

- $\dim(S) \leq \dim(R)$: pick a chain of prime ideals $q_0 \subsetneq q_1 \subsetneq \cdots q_n \subset S$. Then $p_i = q_i \cap R$ forms a chain of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots p_n$ by point 3 of the Going up theorem.

- $\dim(S) \geq \dim(R)$: pick a chain of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots p_n \subset S$. By point 1 of the going up theorem, we can find $q_0 \subset S$ such that $q_0 \cap R = p_0$. By point (2), we can find q_1 such that $q_0 \subset q_1$ with $q_1 \cap R = p_1$, with $q_0 \subsetneq q_1$ necessarily. Continuing like this, we obtain a chain of length n of prime ideals of S.

□

Theorem 7. *Let F be any field. Then $\dim(F[x_1, \cdots, x_n]) = n$.*

Proof. We prove this by induction on n. If $n = 0$, the result is trivially true. Now, we know that $\dim(F[x_1, \cdots, x_n]) \geq n$ since

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \cdots, x_n)$$

is a chain of prime ideals of length n. Take now any chain of prime ideals

$$p_0 \subsetneq p_1 \subsetneq p_2 \cdots \subsetneq p_r;$$

we want to show that $r \leq n$. We can make this chain longer if we can:

- If $p_0 \neq (0)$ then we can add (0) at the beginning: so we can assume $p_0 = (0)$ and that the chain looks like $(0) \subsetneq p_1 \cdots$;
- If p_1 is not principal, take $f \in p_1 \setminus 0$. Since $F[x_1, \cdots, x_n]$ is a UFD, we can pick a prime factor s of f: then (s) is a prime ideal and $(s) \subset p_1$.

Due to these considerations, we can assume that our chain looks like

$$(0) \subsetneq (s) \subsetneq p_2 \cdots \subsetneq p_r,$$

where $s \in F[x_1, \cdots, x_n]$ is irreducible. Now, up to reordering the coordinates, we can assume that $s \notin F[x_1, \cdots, x_{n-1}]$.

Let $R = F[x_1, \cdots, x_n]/(s)$ and let $\bar{x}_i$ be the image of x_i in R. We claim that $\bar{x}_1, \cdots, \bar{x}_{n-1}$ are algebraically independent. Suppose not, then there is a polynomial $P \in F[x_1, \cdots, x_{n-1}]$ such that $P(\bar{x}_1, \cdots, \bar{x}_{n-1}) = 0$. This means that $P \in (s)$, which is impossible unless $P = 0$.

So, we have an inclusion $F[\bar{x}_1, \cdots, \bar{x}_{n-1}] \subset R$ where the former is a polynomial algebra. We do not know that this ring extension is integral, though, so we cannot use the induction step and deduce that R has dimension $n-1$. On the other hand, we know that the induced extension on fraction fields $\mathrm{Frac}(F[\bar{x}_1, \cdots, \bar{x}_{n-1}]) \subset \mathrm{Frac}(R)$, because $\mathrm{Frac}(R)$ is generated by $\bar{x}_n$. Therefore we have $\mathrm{tr.deg.}(\mathrm{Frac}(R)) = n-1$. Also, we can use Noether normalization and find another polynomial algebra $S = F[t_1, \cdots, t_d]$ such that $S \subset R$ is a finite ring extension. Again, since $\mathrm{Frac}(S) \subset \mathrm{Frac}(R)$ is an algebraic field extension, this shows that they have the same transcendental degree, i.e., that $d = n-1$.

But by induction $\dim(S) = n-1$, and since $S \subset R$ is integral, we also have that $\dim(R) = n-1$.

To conclude the argument, we now note that the chain

$$(0) \subsetneq (s) \subsetneq p_2 \cdots \subsetneq p_r$$

yields a chain on R of the form

$$(0) \subsetneq p_2/(s) \cdots \subsetneq p_r/(s),$$

and hence $r - 1 \leq n - 1$, which proves the result. □

We are finally able to prove

Theorem 8. *Let F be a field and let $R = F[x_1, \cdots, x_n]/I$ be an integral domain. Then $\dim(R) = \mathrm{tr.deg.}_F(\mathrm{Frac}(R))$.*

Proof. By Noether normalization we can find a polynomial algebra $S = F[t_1, \cdots, t_d]$ and an integral extension $S \subset R$. Now $\dim(R) = \dim(S) = d$ where the last equality follows from the previous proposition and $\mathrm{tr.deg.}_F(\mathrm{Frac}(R)) = \mathrm{tr.deg.}_F(\mathrm{Frac}(S)) = d$. □