Lecture 12

In this lecture we introduce localization of rings (and modules), which is one of the most important tool in commutative algebra and algebraic geometry. One can motivate it in a variety ways.

1. Constructing Zariski open subsets: let K be an algebraically closed field, and consider $K^n = \mathrm{m}-\mathrm{Spec}(K[x_1, \cdots, x_n]$ endowed with the Zariski topology. We know that we can interpret the elements of $R = K[x_1, \cdots, x_n]$ as regular functions on K^n. Now, an element of $F = \mathrm{Frac}(R)$ can be interpreted as a meromorphic (or rational) function on K^n: take $f/g \in F$ and let $V(g) \subset K^n$ the vanishing locus of g. Then $V(g) \subset K^n$ is a Zariski closed subset by definition, and its complement $D(g) = K^n \setminus V(g)$ is a Zariski open. Note also that $V(g)$ has dimension $n-1$ (if $g \notin K$). This means that $D(g) = K^n \setminus V(g)$ is big: for instance if $n = 1$ then $D(g)$ corresponds to the whole K minus finitely many points, and in general, for any $g_1, g_2 \in R$ we have $D(g_1) \cap D(g_2) = D(g_1 g_2) \neq \emptyset$. The element f/g can now be considered as a regular function $D(g) \to K$ sending $x \mapsto f(x)/g(x)$. The upshot of this is that we can interpret the elements of F as functions which are regular on some Zariski open subset of K^n. The (vague) question now is: is there a ring S whose $\mathrm{m}-\mathrm{Spec}$ gives precisely $D(g)$? Using the philosophy that algebraic varieties should correpond to their ring of functions, to find such ring, we should understand regular functions $D(g) \to K$. Now an element of R gives by restriction a regular function on $D(g)$, so we must get an inclusion $R \subset S$. Moreover, by what we said before, any meromorphic (or rational) function on $D(g)$ should give a rational function on K^n and viceversa, which tells us that $R \subset S \subset F$. In fact, the answer is $S = \{f/g^k : f \in R \text{ and } k \geq 0\}$. Note that this is a subsring of F, which we obtained from R by forcing g to become invertible. This is what localization does: it forces certain elements of the ring to become invertible.

2. Another fact that will become apparent sooner or later is that a ring is easier to study the fewer ideals it has. For instance, fields are the easiest rings, because only (0) is a prime ideal. Similarly, local rings are easier study than general rings because they have a unique maximal ideal. Localization allows us to obtain local rings out of rings. The geometric interpretation of this is that we focus our attention to what happens around a point. As an example, consider the point $P = (0,0) \in K^2$. Now, we consider the subring $S \subset F = \mathrm{Frac}(K[x_1, x_2])$ given by rational functions which are well-defined at $(0,0)$, that is, $S = \{f/g \in F : g(0,0) \neq 0\}$. This is easily seen to be a ring, and we shall check later that

this is also a local ring, where the unique maximal ideal is given precisely by (x_1, x_2), i.e., the maximal ideal of $K[x_1, x_2]$ corresponding to P.

Another example is given by $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \colon (b,p) = 1\}$. This is also a local ring whose only maximal ideal is given by (p).

Definition 1. Let R be a ring. A multiplicatively closed subset $T \subset R$ is a subset such that $1 \in T$ and for every $t, s \in T$ we have $st \in T$.

Example.

1. If $t \in R$ then we can take $T = \{1, t, t^2, t^3, \cdots\}$.
2. If $p \subset R$ is a prime ideal, then $R \setminus p$ is a multiplicatively closed subset (in fact, and ideal $I \subset R$ is prime if and only if $R \setminus I$ is a multiplicatively closed subset).

Those two examples will correspond respectively to the two motivational examples above.

Given a ring R and a multiplicatively closed subset $T \subset R$, localization allows us to build a new ring $T^{-1}R$ in which all the elements of T become invertible, in a universal way:

Theorem 2. *Let R and T be as above. There exists a ring $T^{-1}R$ together with a map $\iota \colon R \to T^{-1}R$ which satisfies the following unviersal property:*

- *for any $t \in T$ the image $\iota(t)$ is invertible in $T^{-1}R$;*
- *If $f \colon R \to S$ is a ring map such that for any $t \in T$ we have that $f(t)$ is invertible in S, then there is a unique map $\tilde{f} \colon T^{-1}R \to S$ and a factorization*

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[dr, "f"'] & T^{-1}R \arrow[d, "\tilde{f}"] \\ & S \end{tikzcd}$$

The ring $T^{-1}R$ is called the localization of R at T.

Proof. Let us show the proof in the easy case when $T = \{1, t, t^2, \cdots\}$ for $t \in R$. In this case, we can construct the ring $T^{-1}R$ quite easily: we simply put $T^{-1}R = R[x]/(xt-1)$. Note what we are doing: we are first constructing the polynomial ring $R[x]$, and then we are forcing the relation $xt = 1$: in this way, the image of x in $R[x]/(xt-1)$ becomes an inverse of t. We show now that $R \to R[x]/(xt-1)$ satisfies the universal property: pick any ring map $f \colon R \to S$ such that $f(t)$ is invertible, i.e., there is (a necessarily unique) $s \in S$ such that $sf(t) = 1$. Note that then also $f(t^n) = f(t)^n$ is invertible for every $n \geq 0$, with inverse s^n. Now consider the map $\hat{f} \colon R[x] \to S$ which sends $P(x) = r_0 + r_1 x + \cdots r_n x^n$ to $f(r_0) + f(r_1)s + \cdots f(r_n)s^n$. We only need to check that $(xt-1) \subset \ker(\hat{f})$. But we compute $\hat{f}(tx-1) = f(t)s - 1 = 0$.

In fact, we can replicate this construction for any multiplicatively closed subset $T \subset R$, but we need to use polynomial rings in infinitely many variables. The argument is explained in Patakfalvi's notes.

The problem with this proof is that, for instance, it is hard to characterise the kernel of the natural map $R \to R[x]/(xt-1)$. The classical proof of the theorem uses calculus of fractions, and it is more intuitive although one has more details to check. We would like to create a new ring whose elements are of the form r/t for $r \in R$ and $t \in T$. One begins by considering the set $\{(r,t)\colon r \in R, t \in T\}$. One then defines addition and multiplication by emulating the usual formulas for fractions: $(r_1,t_1) + (r_1,t_2) = (r_1t_2+r_2t_1, t_1t_2)$ and $(r_1,t_1)(r_2,t_2) = (r_1r_2, t_1t_2)$. Next, one needs to put an equivalence relation on this set to take care of pairs which yield the same fraction. As a first attempt, one can say that two pairs (r_1,t_1) and (r_2,t_2) are equivalence if $r_1t_2 - r_2t_1 = 0$. This indeed yield an equivalence relation if R is a domain, but in general it may fail to be transitive, due to the presence of zero divisors in the multiplicative set T.

The solution is the following: we say that (r_1,t_1) is equivalent to (r_2,t_2) if there is some $u \in T$ such that $u(r_1t_2 - r_2t_1) = 0$. Intuitively, since we are forcing all the elements of T to become invertible, if $u(r_1t_2 - r_2t_1) = 0$ then also $(r_1t_2 - r_2t_1)$ must be zero in the localization $T^{-1}R$. One checks that this gives an equivalence relation, and that the operations defined above respect this equivalence relation (we leave all these easy verifications to the student). So, by taking equivalence classes, we obtain the ring $T^{-1}R$. We denote the equivalence class of (r,t) in $T^{-1}R$ by r/t. The map $\iota\colon R \to T^{-1}R$ is then given by $r \mapsto r/1$.

To check the universal property, we first show that for any $t \in T$ we have that $\iota(t)$ is invertible. We check: $\iota(t) = t/1$ and $t/1 \cdot 1/t = t/t = 1/1$. Now, consider any map $f\colon R \to S$ is such that $f(t)$ is invertible for every $t \in T$, then we define a map $\tilde{f}\colon T^{-1}R \to S$ by sending r/t to $f(r)f(t)^{-1}$. This well-defined, because if $r_1/t_1 = r_2/t_2$ then there is $u \in T$ such that $u(r_1t_2 - r_2t_1) = 0$, so $f(u)(f(r_1)f(t_2) - f(r_2)f(t_1)) = 0$ in S. But since $f(u)$ is invertible in S, this means that $f(r_1)f(t_2) - f(r_2)f(t_1) = 0$ and hence that $f(r_1)f(t_1)^{-1} = f(r_2)f(t_2)^{-1}$. The fact that $\tilde{f}$ is a map of ring is also easy to check. For instance we have $\tilde{f}(r_1/t_1 + r_2/t_2) = \tilde{f}((r_1t_2 + r_2t_1)/(t_1t_2)) = (f(r_1)f(t_2) + f(r_2)f(t_1))(f(t_1)^{-1}f(t_2)^{-1}) = f(r_1)f(t_1)^{-1} + f(r_2)f(t_2)^{-1}$. Multiplication can be checked in an analogous way. $\square$

Using the last description, it is easy to determine the kernel of $\iota\colon R \to T^{-1}R$:

Proposition 3. *We have*

$$\ker \iota = \bigcup_{u \in T} \mathrm{Ann}_R(u).$$

Here, $\mathrm{Ann}_R(u) = \{r \in R\colon ru = 0\}$ is the annihilator of u in R. This is always an ideal. Note also that the union of ideals is in general never an ideal. On the other hand, it can be easily checked that for any multiplicatively closed subset we have that $\bigcup_{u \in T} \mathrm{Ann}_R(u)$ is always an ideal.

Proof. To determine the kernel we simply have to determine for which elements $r \in R$ we have $r/1 = 0/1$ in $T^{-1}R$. This is equivalent that there is $u \in T$ such that $ur = 0$, so $r \in \mathrm{Ann}_R(u)$ for some $u \in T$. $\square$

In particular, we have

1. If $0 \in T$ then $T^{-1}R = 0$, the zero ring (where $1 = 0$).
2. If T contains no zero divisors, then ι is injective. This happens automatically if R is a domain.
3. If R is a domain, we also have a natural injection $T^{-1}R \subset \mathrm{Frac}(R)$ whenever $0 \notin T$. In fact, the natural injection $R \to \mathrm{Frac}(R)$ sends every element of T to an invertible element of $\mathrm{Frac}(R)$. So by the universal property of the localization we obtain a map $T^{-1}R \to \mathrm{Frac}(R)$. This is clearly injective. From this it follows that if R is a domain then also $T^{-1}R$ is a domain, and it can be identified with the subring $\{r/t \in \mathrm{Frac}(R) \colon r \in R \text{ and } t \in T\}$.

Finally, we can also localize R-modules. The construction is basically the same.

Proposition 4. *Let M be a R-module and $T \subset R$ be a multiplicatively closed subset. Then, there is a R-module $T^{-1}M$ and a morphism of R-modules $\iota \colon M \to T^{-1}M$ such that:*

1. *for any $t \in T$, the multiplication by t map $t \colon T^{-1}M \to T^{-1}M$ is an isomorphism;*
2. *If $\phi \colon M \to N$ is a morphism of R-modules such that for any $t \in T$, the multiplication by t map $t \colon N \to N$ is an isomorphism, there is a unique morphism $\tilde{\phi} \colon T^{-1}M \to N$ such that the following commutes:*

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "\phi"'] & T^{-1}M \arrow[d, "\tilde{\phi}"] \\ & N \end{tikzcd}$$

One constructs $T^{-1}M$ using again calculus of fractions. That is, $T^{-1}M$ is the set of equivalence classes of pairs (m, t) for $m \in M$ and $t \in T$, where two such pairs (m_1, t_1) and (m_2, t_2) are equivalent if there is $u \in T$ such that $u(m_1t_2 - m_2t_2) = 0$. Addition is defined in the usual manner, and the R-module structure $R \times T^{-1}M \to T^{-1}M$ is given by $(r, m/t) \mapsto rm/t$.

Two things are important to know: the first is that $T^{-1}M$ is naturally a $T^{-1}R$-module. The second, from which the first also follows, and which is part of the exercises, is that there is a natural isomorphism $T^{-1}M \cong M \otimes_R T^{-1}R$.

Definition 5. This is just notation: if R is a ring and $t \in R$, then we denote by R_t the localization of R at $T = \{1, t, t^2, \cdots\}$. If $p \subset R$ is a prime ideal, then one denotes by R_p the localization of R at $T = R \setminus p$.

0.1 Behaviour of ideals under localization

Now we have come to arguably the most important result concerning localization. Consider $f \colon R \to S$ any ring morphism. If $J \subset S$ is an ideal, then $J^c := f^{-1}(J) \subset R$ is also an ideal. This is called the contraction of J along f.

If $I \subset R$ is an ideal, then $f(I) \subset S$ is not in general an ideal. On the other hand, we define the extension of I along f as $I^e \subset S$ the ideal generated by $f(I)$, i.e., $I^e = f(I) \cdot S$. So if $I = (r_1, \cdots, r_n)$ then $I^e = (f(r_1), \cdots, f(r_n))$.

We already know that contraction respects prime ideals. This was part of the exercises. On the other hand, extension does not preserve primality. Also the following containments are obvious: $(J^c)^e \subset J$ and $I \subset (I^e)^c$.

Theorem 6. *Let $T \subset R$ be a multiplicatively closed subset, and let $\iota \colon R \to T^{-1}R$ be the localization map.*

1. *For any ideal $J \subset T^{-1}R$ we have $(J^c)^e = J$;*
2. *For $I \subset R$ we have $(I^e)^c = \bigcup_{t \in T}(I \colon t) = \bigcup_{t \in T}\{r \in R \colon rt \in I\}$;*
3. *For $I \subset R$ we have that $I^e = (1)$ if and only if $I \cap T \neq \emptyset$;*
4. *For any $p \subset R$ prime ideal such that $p \cap T = \emptyset$ we have: p^e is a prime ideal of $T^{-1}R$ and $(p^e)^c = p$.*

Proof.
1. we know already that $(J^c)^e \subset J$. Now, take any $r/t \in J$: then also $(t/1)(r/t) = r/1 \in J$. But then $r \in J^c$ and therefore $r/t \in (J^c)^e$.
2. I^e is the ideal of $T^{-1}R$ generated by $\iota(I)$. So, a random element of I^e has the form $\sum_{\text{finite}} \frac{r_k}{t_k} \cdot \frac{i_k}{1}$ where $r_k \in R$, $t_k \in T$ and $i_k \in I$. But a simple computation with fractions then shows that every element of I^e has the form i/t for $i \in I$ and $t \in T$. Now, an element $r \in R$ belongs to $(I^e)^c$ if and only if there is $i/t \in I^e$ such that $r/1 = i/t$. That is, if and only if there is $i \in I$, $u, t \in T$ such that $u(rt - i) = 0$. If this last equality is true, then $r(ut) \in I$, so $r \in (I \colon ut)$. Similarly, if $r \in (I \colon t)$ for some $t \in T$, then $rt = i \in I$ and therefore $r/1 = i/t$.
3. We have that $I^e = (1)$ if and only $i/t = 1/1$ for some $i \in I$, $t \in T$. This means that there is $u \in T$ such that $(i - t)u = 0$. But then $iu = tu$ which shows that $I^e = (1)$ if and onl $I \cap T \neq \emptyset$.
4. Let us first show that p^e is a prime ideal if $p \cap T = \emptyset$. Assume that there are $r_1/t_1, r_2/t_2 \in T^{-1}R$ such that $r_1r_2/t_1t_2 \in p^e$. But then also $(r_1/1)(r_2/1) \in p^e$, which means that there are $p \in p$, $u, t \in T$ such that $u(tr_1r_2 - p) = 0$ in R which implies that $(ut)(r_1r_2) \in p$. Now $ut \notin p$ by assumption, so necessarily $r_1r_2 \in p$. So either r_1 or r_2 is in p, which shows that either r_1/t_1 or r_2/t_2 is in p^e.

   To show that $(p^e)^c = p$ we use the formula from point 2. It is enought to prove that for any $t \in T$ we have $(p \colon t) = p$. But $(p \colon t) = \{r \in R \colon rt \in p\}$ by definition, and since $t \notin p$ for every $t \in T$, we conclude.

□

In particular, we have

Corollary 7. *Contraction and extension gives a one-to-one correspondence*

$$\{\textit{prime ideals of } T^{-1}R\} \longleftrightarrow \{\textit{prime ideals } p \textit{ of } R \textit{ such that } p \cap T = \emptyset\}.$$

In particular, the map $\mathrm{Spec}(T^{-1}R) \to \mathrm{Spec}(R)$ *is injective, and its image corresponds to all* $p \in \mathrm{Spec}(R)$ *such that* $p \cap T = \emptyset$.

Proof. If p is a prime ideal of $T^{-1}R$, then p^c is a prime ideal of R. Since $(p^c)^e = p$ by point 1, we have that $p^c \cap T = \emptyset$ necessarily by point 3.

If $p \subset R$ is a prime ideal such that $p \cap T = \emptyset$ then p^e is prime by point 4, and $(p^e)^c = p$ again by point 4. □

Corollary 8. *Let* $p \subset R$ *be a prime ideal, and let* $T = R \setminus p$. *Then* $T^{-1}R$ *is a local ring, with maximal ideal given by* p^e.

Proof. Clearly $p \cap T = \emptyset$ so p^e is a prime ideal of $T^{-1}R$. But now any element of $T^{-1}R \setminus p^e$ is invertible, so p^e has to be maximal as well. □

This shows that in general localization does not respect maximal ideals. On the other hand, we have the following consequence of Nullstellensatz:

Proposition 9. *Let* $R = K[x_1, \cdots, x_n]$ *with* K *algebraically closed. Let* $p \subset R$ *be a prime ideal, and assume that* $f \notin p$ *for some* $f \in R$. *Then, there is a maximal ideal* $p \subset m \subset R$ *such that* $f \notin m$.

Proof. Assume that $f \in m$ for every maximal ideal containing p. This means that f is constantly zero on $V(p)$, so that $f \in I(V(p))$. By nullstellensatz, $I(V(p)) = \sqrt{p} = p$ hence $f \in p$, contraddiction. □

In particular, to reconnect to our original motivation

Corollary 10. *Let* K *be algebraically closed, and let* $g \in K[x_1, \cdots, x_n]$. *Then for any maximal ideal* $m \subset K[x_1, \cdots, x_n]_g$ *we have that* m^c *is maximal, and contraction-extension induces a one-to-one identification*

$$\{\textit{maximal ideals of } K[x_1, \cdots, x_n]_g\} \longleftrightarrow \{(x_1 - c_1, \cdots, x_n - c_n) \textit{ such that } g(c_1, \cdots, c_n) \neq 0\}.$$

Proof. If m is a maximal ideal of $K[x_1, \cdots, x_n]_g$ then m^c is a prime ideal of $K[x_1, \cdots, x_n]$. Moreover, $g \notin m^c$, for otherwise $m = (m^c)^e = (1)$ which is a contradiction. But if m^c is not maximal, then there is a maximal ideal $m^c \subset n$ such that $g \notin n$ by the previous proposition. Hence $n^e \neq 1$ and therefore $m = n^e$ by maximality. But then $m^c = (n^e)^c = n$. The other inclusion is easy to verify. □

This shows that in particular $\mathrm{m} - \mathrm{Spec}(K[x_1, \cdots, x_n]_g)$ is naturally identified to the Zariski open $D(g)$.