Lecture 9

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0.1 Some comments on the previous lecture

In the last lecture we introduced algebraic subsets of K^n, for K an algebraically closed field. Hilbert Nullstellensatz says that there is a one-to-one correspondence between algebraic subsets of K^n and radical ideals $I \subset K[x_1, \cdots, x_n]$. Now, pick a radical ideal I and consider the corresponding algebraic subset $V(I) \subset K^n$. One would like to define polynomial functions $f\colon V(I) \to K$. This is done as follows. First, note that any $f \in K[x_1, \cdots, x_n]$ can be naturally considered as a polynomial function

$$f\colon K^n \to K \tag{0.1}$$
$$(c_1, \cdots, c_n) \mapsto f(c_1, \cdots, c_n). \tag{0.2}$$

Then, to obtain a polynomial function on $V(I)$, we simply restric the function above to $V(I)$

Definition 1. A polynomial function $V(I) \to K$ is any function of the form $f_{|V(I)}$ where $f \in K[x_1, \cdots, x_n]$.

Now, observe that since K is a ring, the set of polynomial functions on $V(I)$ is also a ring (where addition and multiplication are defined on the target). In fact, it is easy to see that the ring of polynomial functions on $V(I)$ is precisely $K[x_1, \cdots, x_n]/I$ (why?). Thus, we interpret $R = K[x_1, \cdots, x_n]/I$ as the ring of functions on $V(I) = m - \mathrm{Spec}(R)$ (where this equality follows from the previous lecture, and it is actually an isomorphism of topological spaces, where we endow both of them with the Zariski topology).

Example. Suppose that I is radical, but not prime, and still write $R = K[x_1, \cdots, x_n]/I$. Then, there are $f, g \in R$ such that $f, g \neq 0$ but $fg = 0$. This means that we have a topological space $V(I)$ and continuos functions $f, g\colon V(I) \to K$ which are non-zero, but whose product is zero. Now, $\{f = 0\}, \{g = 0\} \subset V(I)$ are closed subsets by continuity. Moreover, $\{f = 0\}, \{g = 0\} \subsetneq V(I)$ because neither of them are constantly zero by assumption. On the other hand, $\{f = 0\} \cup \{g = 0\}$ because their product is zero. This means that when I is not prime, the topological space $V(I)$ is not irreducible (as also explained in the previous lecture).

1 Dimension theory

In this section, we would like to define the notion of dimension of $R = K[x_1, \cdots, x_n]/I$ where I is a prime ideal (and, in particular, radical). The reason why we do this only for prime ideals is clear: if I is not prime, then $V(I)$ is not irreducible, and can be written as a union of closed subsets, which may have different dimensions. For instance, we may consider $K[x, y]$ and $I = (x(x-1), x(y-1)) = (x)(x-1, y-1)$. Then $V(I)$ corresponds in K^2 to the y-axis and the point $(1, 1)$. Intuitively, the former has dimension 1, and the latter has dimension 0. Our first definition of dimension will work well only for finitely generated K-algebras, and uses the concept of transcendental degree of field extension.

1.1 Transcendental degree

Let $F \subset L$ be a field extension, and let $A \subset L$ be any subset.

Definition 2. We say that A is algebraically independent over F if for any distinct elements $a_1, \cdots, a_m \in A$ and any polynomial $f(x_1, \cdots, x_n) \in F[x_1, \cdots, x_n]$ we have that $f(a_1, \cdots, a_n) \neq 0$.

In general, for any subset $A \subset L$ we define $F \subset F[A] \subset L$ as the sub F-algebra generated by A, and by $F \subset F(A) \subset L$ as the subfield generated by A. We always have $F(A) = \mathrm{Frac}(F[A])$ (note that $F[A]$ is a domain, why?). Moreover, any $f \in F[A]$ can be written as $f(a_1, \cdots, a_m)$ for some $f \in F[x_1, \cdots, x_m]$ and $a_1, \cdots, a_m \in A$.

Definition 3. Let $F \subset L$ be a field extension. We say that $A \subset L$ is a transcendental basis for L over F if:

1. A is algebraically independent over F;
2. L is algebraic over $F(A)$ (i.e., the extension $F(A) \subset L$ has finite degree).

Now, we prove three lemmas. We fix $F \subset L$ field extension.

Lemma 4. *Let $a_1, \cdots, a_m, b \in L$. Assume that b is algebraic over $F(a_1, \cdots, a_m) \subset L$. Then, there is a polynomial $f(x_1, \cdots, x_m, y) \in F[x_1, \cdots, x_m, y]$ such that*

1. *$f(a_1, \cdots, a_m, y) \in F(a_1, \cdots, a_m)[y]$ is not zero;*
2. *$f(a_1, \cdots, a_m, b) = 0$.*

Proof. Since b is algebraic over $F(a_1, \cdots, a_m)$, we can consider its minimal polynomial

$$P(y) = y^n + c_{n-1}y^{n-1} + \cdots c_1 y + y \in F(a_1, \cdots, a_m)[y].$$

Now, any $c \in F(a_1, \cdots, a_m)$ can be written as

$$c = \frac{g(a_1, \cdots, a_m)}{h(a_1, \cdots, a_m)}$$

for some $g, h \in F[x_1, \cdots, x_m]$. So, we write $c_i = \frac{g_i(a_1, \cdots, a_m)}{h_i(a_1, \cdots, a_m)}$. Note that $h_i(a_1, \cdots, a_m) \neq 0$ for every i. Now, we delete denominators, and we let

$$Q(y) := \prod_i h_i(a_1, \cdots, a_m) \cdot P(y) = \prod_i h_i(a_1, \cdots, a_m) \cdot (\sum_j c_j y^j) =$$

$$= \sum_j \left(g_j(a_1, \cdots, a_m) \prod_{i \neq j} h_i(a_1, \cdots, a_m) \right) y^j$$

Note that $Q(y) \neq 0$ and that $Q(b) = 0$. Now, consider

$$f(x_1, \cdots, x_n, y) \coloneqq \sum_j \left(g_j(x_1, \cdots, x_m) \prod_{i \neq j} h_i(x_1, \cdots, x_m) \right) y^j;$$

then $f(a_1, \cdots, a_m, y) = Q(y) \neq 0$ and $f(a_1, \cdots, a_m, b) = 0$ which proves the lemma.

□

Lemma 5 (Exchange lemma)**.** *Assume that $b \in L$ is algebraic over $F(a_1, \cdots, a_m)$ but not algebraic over $F(a_1, \cdots, a_{m-1})$. Then, a_m is algebraic over $F(a_1, \cdots, a_{m-1}, b)$.*

Proof. We use the previous lemma and we find a polynomial $f(x_1, \cdots, x_m, y)$ such that $f(a_1, \cdots, a_m, y) \neq 0$ and $f(a_1, \cdots, a_m, b) = 0$. Consider now the polynomial

$$0 \neq f(a_1, \cdots, a_{m-1}, x_m, y) \in F(a_1, \cdots, a_{m-1})[x_m, y]$$

and write it as

$$f(a_1, \cdots, a_{m-1}, x_m, y) = \sum_{i,j} f_{ij} x_m^i y^j, \quad f_{ij} \in F(a_1, \cdots, a_{m-1}).$$

We now rearrange the terms and write it as a polynomial in x_m:

$$= \sum_i \left(\sum_j f_{ij} y^j \right) x_m^i.$$

Since this is not zero, at least one of its coefficients is not zero, i.e., there is i such that $\sum_j f_{ij} y^j \neq 0$. Moreover, $\sum_j f_{ij} b^j \neq 0$ as well, because otherwise b would be algebraic over $F(a_1, \cdots, a_{m-1})$, which contraddicts our assumption. From this it follows that

$$0 \neq P(x_m) \coloneqq \sum_i \left(\sum_j f_{ij} b^j \right) x_m^i \in F(a_1, \cdots, a_{m-1}, b)[x_m].$$

But $P(a_m) = 0$, so this proves the lemma. □

Lemma 6. *Let $A = \{a_1, \cdots, a_n\}, B = \{b_1, \cdots, b_m\} \subset L$. Assume that*

1. *A is algebraically independent over F;*
2. *A is algebraic over $F(B)$.*

Then $n \leq m$.

Proof. Up to reordering, write $A \cap B = \{a_1, \cdots, a_r\} = \{b_1, \cdots, b_r\}$. We can consider the same statement for $F' = F(A \cap B)$ and $A' = \{a_{r+1}, \cdots, a_m\}$ and $B' = \{b_{r+1}, \cdots, b_n\}$. In fact, clearly A' is algebraic independent over F' and algebraic over $F'(B')$. If the statement is true, then we get $n - r \leq m - r$ i.e. $n \leq m$. So we can assume that $A \cap B =$ and prove the statement by induction on $|A|$. The case $|A| = 0$ is obvious.

Choose $C \subset \{b_1, \cdots, b_m\}$ minimal with respect to $|C|$ such that a_1 is algebraic over $F(C)$. Such C must exist since A is algebraic over $F(B)$.

1. $C \neq$, for otherwise a_1 would be algebraic over F, constraddicting that A is algebraically independent over F;

2. If $b_i \in C$, then b_i cannot be algebraic over F, for otherwise $C' = C \setminus \{b_i\}$ would still be such that a_1 is algebraic over $F(C')$, contraddicting minimality.

Now, pick any $b_i \in C$. So, a_1 is algebraic over $F(C)$ but not algebraic over $F(C \setminus \{b_i\})$, again by minimality. Let $C' = (C \setminus \{b_i\}) \cup \{a_1\}$. By the exchange lemma, b_i is algebraic over $F(C')$. Now, consider a new $B' = (B \setminus \{b_i\}) \cup \{a_1\}$. Note that $|B'| = |B|$ because $a_1 \neq B$ since $A \cap B = \emptyset$. On the other hand, A is still algebraic over $F(B')$, because b_i is algebraic over $F(B')$ and hence $F(B)$ is algebraic over $F(B)$. But now $A \cap B' = \{a_1\}$. So we can use the argument at the beginning to reduce to the case where $A = \{a_2, \cdots, a_n\}$ and $B = \{b_1, \cdots, \hat{b_i}, \cdots, b_n\}$ and use induction.

□

Corollary 7. *If $F \subset L$ has a finite transcendental basis, then all transcendental basis have the same cardinality.*

This is true also if the cardinality is not finite, but we won't prove it.

Definition 8. The transcendental degree $\mathrm{tr.deg.}_F(L)$ of $F \subset L$ is the cardinality of a transcendental basis of L over F.

Lemma 9. *Assume that $R = K[x_1, \cdots, x_n]/I$ is a domain. Then $\mathrm{Frac}(R)$ has a finite transcendental basis over K.*

Proof. Note that R is generated, as a K-algebra, by the images $\bar{x}_i$ of the variables $x_i \in K[x_1, \cdots, x_n]$. So, $\mathrm{Frac}(R)$ is also generated as a field over K by $A_0 = \{\bar{x}_1, \cdots, \bar{x}_n\}$. Now, either A_0 is algebraic independent, in which case A_0 is a transcendental basis of $\mathrm{Frac}(R)$ over K, or there is an element $\bar{x}_i \in A_0$ (which we may assume wlog to be $\bar{x}_1$) that is algebraic over $A_1 = \{\bar{x}_2, \bar{x}_3, \cdots, \bar{x}_n\}$. In this case, $K(A_1) \subset \mathrm{Frac}(R)$ is an algebraic extension. Now, again, wither all the elements of A_1 are algebraic independent over K, so that A_1 is a transcendental basis over K, or there is an element $\bar{x}_i \in A_1$, which we may suppose to be $\bar{x}_2$ up to reordering, which is algebraic over $A_2 = \{\bar{x}_3, \bar{x}_4, \cdots, \bar{x}_n\}$. Note that again $K(A_2) \subset \mathrm{Frac}(R)$ is algebraic. Keeping on going like this, we arrive at a certain A_k with $k \leq n$ such that all the elements of A_k are algebraically independent and $K(A_k) \subset \mathrm{Frac}(R)$ is algebraic. Thus A_k must be a transcendental basis of $\mathrm{Frac}(R)$ over K. □

Example. 1. The transcendental degree of $F(x_1, \cdots, x_n)$ over F is clearly n;

2. Let us compute the transcendental degree of the fraction field of $K[x, y]/(x^2 - y^3)$. Let us check that $(x^2 - y^3)$ is prime, first of all. Since $K[x, y]$ is a UFD, it is enough to show that $x^2 - y^3$ is irreducible. Now, we can see this as a polynomial over $K(y)$. Since $\sqrt{y^3}$ does not belong to $K(y)$, we see that the only possible factorizations are of the form $P(y) \cdot Q(y, x)$. But this does not make sense in $K[x, y]$ unless P is constant. So $R = K[x, y]/(x^2 - y^3)$ is a domain, and $F \operatorname{Frac}(R)$ is a field extension of K. Let $\bar{x}, \bar{y}$ be the images of x, y respectively in R. Now, we have $K \subset K(\bar{x}) \subset F$. Clearly, the last extension is finite: F is generated by $\bar{x}$ and $\bar{y}$ and $\bar{y}$ is algebraic over $K(\bar{x})$ because it satisfies the equation $t^3 - \bar{x}^2 = 0$. Finally, we want to check that $\bar{x}$ is algebraically independent over K. But assume that there are $c_i \in K$ such that

$$c_0 + c_1\bar{x} + \cdots + c_n(\bar{x})^n = 0.$$

We can clearly assume $c_0 \neq 0$. Now, we lift this equation to $K[x, y]$ and we find

$$c_0 + c_1 x + \cdots + c_n x^n = (x^2 - y^3) \cdot P(x, y).$$

But by putting $x = y = 0$ we get $c_0 = 0$ which is a contraddiction. Thus, $\bar{x}$ is a transcendental basis of $\operatorname{Frac}(R)$ over K.

Definition 10. Let R be a domain of the form $K[x_1, \cdots, x_n]/I$ for some field K. We defined $\dim(R) = \dim(V(I)) = \operatorname{tr.deg.}_K \operatorname{Frac}(R)$.

2 Krull dimension

There is another notion of dimension which works for every ring. The intuitive idea is the following (still K algebraically closed). Consider $f \in K[x_1, \cdots, x_n]$. Then $V(f) \subset K^n$ must have dimension $n-1$, because it is the level set of the (surjective) function $f\colon K^n \to K$ at the point 0. In general $V(f, g) \subset V(f)$ should have dimension $n-2$ (except that in some degenerate cases..), and so on. This suggests, for I a prime ideal, the following definition of dimension:

$$\dim(V(I)) = \text{ length of maximal chain } Z_0 \subsetneq Z_1 \subsetneq \cdots Z_n = V(I),$$

where each $Z_i \subset V(I)$ is irreducible and closed. Using the Nullstellensatz, this translates to ring into the following general defintion:

Definition 11. Let R be a commutative ring. Then, the Krull dimension of R is

$$\sup\{n\colon \exists \text{ a chain of prime ideals } p_0 \subsetneq p_1 \subsetneq p_2, \cdots, \subsetneq p_n \subsetneq R\}.$$

If $p \subset R$ is a prime ideal, its height is defined as

$$\operatorname{ht}(p) = \sup\{n\colon \exists \text{ a chain of prime ideals } p_0 \subsetneq p_1 \subsetneq p_2, \cdots, \subsetneq p_n = p\}.$$

Example. 1. Consider $R = K[x_1, \cdots, x_n]$. Then, we have a chain of prime ideals

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \cdots, x_n).$$

So the Krull dimension of $K[x_1, \cdots, x_n]$ is at least n. Let us show that this chain is maximal, in the sense that if

$$(x_1, \cdots, x_k) \subset p \subset (x_1, \cdots, x_{k+1})$$

is a prime ideal, then necessarily $p = (x_1, \cdots, x_k)$ or $p = (x_1, \cdots, x_{k+1})$. Consider the quotient $K[x_1, \cdots, x_n]/(x_1, \cdots, x_k)$. By renaming the variables, and by using the correspondence between ideals of the quotients and ideals containing $(x_1, \cdots, x_k)$, it is enough to show the following: for any polynomial ring $K[x_1, \cdots, x_n]$, if $p \subset (x_1)$ is a prime ideal, then $p = (x_1)$ or $p = (0)$. Now, to prove this, assume that $p \neq 0$, and take $f \in p$. Since $f \in (x_1)$, we can write $f = x_1^i f_0$ with $f_0 \notin (x_1)$ and $i > 0$. Since p is prime, then either $x_1 \in p$, which then implies $p = (x_1)$ or $f_0 \in p$, which is absurd.

2. The Krull dimension of any field is 0 (so, in algebraic geometry, fields correspond to points);

3. The Krull dimension of a PID is one (why?). So, in algebraic geometry, $\mathbb{Z}$ correspond to a 'curve'.

We will prove in the following lectures the following fundamental theorem:

Theorem 12. *Let $K[x_1, \cdots, x_n]/I$ be a domain. Then, the Krull dimension of R is the same as the transcendental degree of* $\mathrm{Frac}(R)$ *over K.*