Lecture 7

Domenico Valloni

This is our last lecture on homological algebra. We begin by proving the horseshoe lemma introduced in the last lecture.

Notation. *Given a bunch of modules $A_1, \cdots, A_n$ and $B_1, \cdots, B_m$ we shall write an element $f \in \operatorname{Hom}(A_1 \oplus \cdots \oplus A_n, B_1 \oplus \cdots \oplus B_m)$ in matrix form, as follows:*

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}$$

where $f_{ij} \colon A_j \to B_i$ and for $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in A_1 \oplus \cdots \oplus A_n$ we have

$$f \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_{11}(a_1) + \cdots + f_{1n}(a_n) \\ \vdots \\ f_{m1}(a_1) + \cdots + f_{mn}(a_n) \end{pmatrix} \in B_1 \oplus \cdots \oplus B_m$$

Now, we begin the proof of the horseshoe lemma. We begin with a s.e.s. of modules $0 \to A \to B \to C \to 0$ and two projective resolutions $P_\bullet \to A$ and $R_\bullet \to C$. As already said in the last lecture, our aim is to fill the dotted arrows of the following diagram

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_3 \arrow[r, "d_3^A"] \arrow[d] & P_2 \arrow[r, "d_2^A"] \arrow[d] & P_1 \arrow[r, "d_1^A"] \arrow[d] & P_0 \arrow[r, "d_0^A"] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
\cdots \arrow[r, dashed] & P_3 \oplus R_3 \arrow[r, dashed] \arrow[d] & P_2 \oplus R_2 \arrow[r, dashed] \arrow[d] & P_1 \oplus R_1 \arrow[r, dashed] \arrow[d] & P_0 \oplus R_0 \arrow[r, dashed] \arrow[d] & B \arrow[r] \arrow[d, "\psi"] & 0 \\
\cdots \arrow[r] & R_3 \arrow[r, "d_3^C"] & R_2 \arrow[r, "d_2^C"] & R_1 \arrow[r, "d_1^C"] & R_0 \arrow[r, "d_0^C"] & C \arrow[r] \arrow[d] & 0 \\
 & & & & & 0 &
\end{tikzcd}
$$

in such a way that:

1. The resulting diagram becomes commutative;
2. The middle row is a (necessarily projective) resolution of B.

Note. *The vertical maps $P_n \to P_n \oplus R_n$ and $P_n \oplus R_n \to R_n$ are nothing but the inclusion of the first factor and, resplectively, the projection onto the second factor.*

Hence, if we manage to do this, we obtain automatically a short exact sequence of projective resolutions, which is exactly what we need to obtain the long exact sequence of Ext modules as explained in the previous lecture. As in the previous lecture, we put $Q_n \coloneqq P_n \oplus R_n$.

Lemma 1. *Suppose that we are able to find maps $d_n^B \colon Q_n \to Q_{n-1}$ such that the diagram above is commutative and $(Q_\bullet, d_\bullet^B)$ is a complex (i.e., $d_n^B \circ d_{n+1}^B = 0$) and $d_0^B \colon Q_0 \to B$ is surjective. Then, $Q_\bullet \to B$ is automatically a resolution of B.*

Proof. To check that this is a resolution, we only need to prove that all the homology groups of $Q_\bullet$ vanish. Put $P_{-1} = A$, $Q_{-1} = B$ and $R_{-1} = C$. It follows that we have a short exact sequence of complexes $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$. We know that this induces a long exact sequence in cohomology

$$\cdots \to H_n(P_\bullet) \to H_n(Q_\bullet) \to H_n(R_\bullet) \to \cdots$$

and since $H_n(P_\bullet) = H_n(R_\bullet) = 0$ for every n (because those are exact by assumption) we deduce that also $H_n(Q_\bullet) = 0$ for every n, hence that $Q_\bullet$ is exact (and, therefore, a resolution of B). $\square$

Hence it follows that we only need to fill the dotted arrows such that

1. The resulting diagram becomes commutative;
2. The middle row is a complex.

First step: determine d_0^B. Let us look at the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
P_0 \arrow[r, "d_0^A"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
 & B \arrow[d, "\psi"] & \\
R_0 \arrow[ur, dashed, "\tilde{d}_0^C"] \arrow[r, "d_0^C"] & C \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

since R_0 is projective and ψ is surjective, we can fill the dotted arrow and find some $\tilde{d}_0^C \colon R_0 \to B$ making the diagram commutative. Using the matrix notation, we then define $d_0^B = (\phi \circ d_0^A, \tilde{d}_0^C)$. In this way, we clearly get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
P_0 \arrow[r, "d_0^A"] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
P_0 \oplus R_0 \arrow[r, "d_0^B"] \arrow[d] & B \arrow[r] \arrow[d, "\psi"] & 0 \\
R_0 \arrow[r, "d_0^C"] & C \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

and it is easy to check that the middle row is surjective (why?).

Second step: determine d_1^C. Consider now the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
P_1 \arrow[r, "d_1^A"] \arrow[d] & P_0 \arrow[r, "d_0^A"] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
P_1 \oplus R_1 \arrow[r, dashed] \arrow[d] & P_0 \oplus R_0 \arrow[r, "d_0^B"] \arrow[d] & B \arrow[r] \arrow[d, "\psi"] & 0 \\
R_1 \arrow[r, "d_1^C"] \arrow[d] & R_0 \arrow[r, "d_0^C"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

we need to find d_1^B, which again we write in matrix notation

$$d_1^B = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Now, the commutativity of the diagram

$$\begin{tikzcd}
P_1 \arrow[r, "d_1^A"] \arrow[d] & P_0 \arrow[d] \\
P_1 \oplus R_1 \arrow[r, dashed] & P_0 \oplus R_0
\end{tikzcd}$$

forces $g_{11} = d_1^A$ and $g_{21} = 0$ whereas the commutativity of the diagram

$$\begin{tikzcd}
P_1 \oplus R_1 \arrow[r, dashed] \arrow[d] & P_0 \oplus R_0 \arrow[d] \\
R_1 \arrow[r, "d_1^C"] & R_0
\end{tikzcd}$$

forces $g_{22} = d_1^C$. Hence,

$$d_1^B = \begin{pmatrix} d_1^A & g_1 \\ 0 & d_1^C \end{pmatrix}$$

for some $g_1 \colon R_1 \to P_0$. Finally, the condition that $d_0^B \circ d_1^B = 0$ means that (using matrix multiplication)

$$0 = (\phi \circ d_0^A, \tilde{d}_0^C) \begin{pmatrix} d_1^A & g_1 \\ 0 & d_1^C \end{pmatrix} = (0, \phi \circ d_0^A \circ g_1 + \tilde{d}_0^C \circ d_1^C)$$

where the first zero on the right hand side comes from $d_0^A \circ d_1^A = 0$. It follows that we only need to find $g_1 \colon R_1 \to P_0$ such that $\phi \circ d_0^A \circ g_1 + \tilde{d}_0^C \circ d_1^C = 0$. To show that such

g_1 exists, look again at the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
P_1 \arrow[r, "d_1^A"] \arrow[d] & P_0 \arrow[r, "d_0^A"] \arrow[d] & A \arrow[r] \arrow[d, "\phi"] & 0 \\
P_1 \oplus R_1 \arrow[r, dashed] \arrow[d] & P_0 \oplus R_0 \arrow[r, "d_0^B"] \arrow[d] & B \arrow[r] \arrow[d, "\psi"] & 0 \\
R_1 \arrow[r, "d_1^C"] \arrow[d] & R_0 \arrow[r, "d_0^C"'] \arrow[ur, "\tilde{d}_0^C"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

The first thing to notice is that $-\tilde{d}_0^C \circ d_1^C$ lands in the image of ϕ. Equivalently, this means that $-\psi \circ \tilde{d}_0^C \circ d_1^C = 0$ which follows from the fact that $\psi \circ \tilde{d}_0^C = d_0^C$. But then, the following composition is a surjection

$$P_0 \xrightarrow{d_0^A} A \xrightarrow{\phi} \mathrm{Im}(\phi)$$

since d_0^A is a surjection. Then, by the projectivity of R_1, we can find $g_1 \colon R_1 \to P_0$ such that $\phi \circ d_0^A \circ g_1 = -\tilde{d}_0^C \circ d_1^C$.

<u>Final step:</u> induction. Suppose now that we have determined $d_n^B = \begin{pmatrix} d_n^A & g_n \\ 0 & d_n^C \end{pmatrix}$ up to some $n > 1$, where $g_n \colon R_n \to P_{n-1}$. We want to find d_{n+1}^B. Using the same reasoning as before, we see that again d_{n+1}^B has the form

$$d_{n+1}^B = \begin{pmatrix} d_{n+1}^A & g_{n+1} \\ 0 & d_{n+1}^C \end{pmatrix}$$

for some $g_{n+1} \colon R_{n+1} \to P_n$. Imposing the condition that $d_n^B \circ d_{n+1}^B = 0$ then yields

$$0 = \begin{pmatrix} d_n^A & g_n \\ 0 & d_n^C \end{pmatrix} \begin{pmatrix} d_{n+1}^A & g_{n+1} \\ 0 & d_{n+1}^C \end{pmatrix} = \begin{pmatrix} 0 & d_n^A g_{n+1} + g_n d_{n+1}^C \\ 0 & 0 \end{pmatrix}$$

so, we win if we can find g_{n+1} such that $d_n^A g_{n+1} + g_n d_{n+1}^C = 0$. Let us look at the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "d_n^A"] & P_{n-1} \arrow[r, "d_{n-1}^A"] & P_{n-2} \arrow[r, "d_{n-2}^A"] & \cdots \\
R_{n+1} \arrow[r, "d_{n+1}^C"'] \arrow[ur, dashed] & R_n \arrow[r, "d_n^C"'] \arrow[ur, "g_n"] & R_{n-1} \arrow[r, "d_{n-1}^C"'] \arrow[ur, "g_{n-1}"] & R_{n-2} \arrow[r] \arrow[ur, "g_{n-2}"] & \cdots
\end{tikzcd}$$

again, to find g_{n+1}, we need to use that R_{n+1} is projective. We can do this as soon as we check that $-g_n d_{n+1}^C \colon R_{n+1} \to P_{n-1}$ lands in the image of d_n^A. But this is equivalent to

$$d_{n-1}^A \circ g_n \circ d_{n+1}^C = 0.$$

By induction, we have that $d^A_{n-1} \circ g_n = -g_{n-1} \circ d^C_n$ and by substituting we get that

$$d^A_{n-1} \circ g_n \circ d^C_{n+1} = -g_{n-1} \circ d^C_n \circ d^C_{n+1} = 0$$

because $d^C_n \circ d^C_{n+1} = 0$. This concludes the proof.

Example. In practice, the long exact sequence of Ext modules is used to compute Ext as follows: suppose that we want to compute $\mathrm{Ext}^i(A, N)$ for some R-modules A, N. One then finds a short exact sequence containing A, e.g.,

$$0 \to A_1 \to A \to A_2 \to 0$$

and such that one known $\mathrm{Ext}^i(A_1, N)$ and $\mathrm{Ext}^i(A_2, N)$. One then uses the long exact sequence to determine $\mathrm{Ext}^i(A, N)$. As a practical example, let us compute $\mathrm{Ext}^i(\mathbb{Z}/n, \mathbb{Z}/m)$ for $R = \mathbb{Z}$ and $n, m > 1$. The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

yields the long exact sequence of Ext groups

$$0 \to \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/m) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/m) \to \mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}/m) \to \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/m) \to \cdots$$

now, since $\mathbb{Z}$ is projective, we know that $\mathrm{Ext}^i(\mathbb{Z}, \mathbb{Z}/m) = 0$ for every $i > 0$. From the exactness of the sequence above, it then follows that also $\mathrm{Ext}^i(\mathbb{Z}/n, \mathbb{Z}/m) = 0$ for $i > 1$. By identifying $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/m) \cong \mathbb{Z}/m$ we then get an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \to \mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m \to \mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}/m) \to 0$$

and hence

1. $\mathrm{Ext}^0(\mathbb{Z}/n, \mathbb{Z}/m) = \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/(n, m)$;
2. $\mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathrm{coker}(\mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m) \cong \mathbb{Z}/(n, m)$;
3. $\mathrm{Ext}^i(\mathbb{Z}/n, \mathbb{Z}/m) = 0$ for $i > 1$.

1 Ext and Yoneda extensions

We retain the usual notation: R is a ring and N, M are modules over it. In this final chapter, we shall give a concrete interpretation of Ext modules in terms of extensions.

Definition 2.

- An extension of N by M is a s.e.s.

$$\mathcal{E} \coloneqq 0 \to N \to E \to M \to 0$$

of R-modules.

- Two extensions $\mathcal{E}$ and $\mathcal{E}'$ are Yoneda equivalent if there is a morphism $f\colon E \to E'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{f} & & \big\downarrow{=} & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

  where the outmost vertical maps are the identity of N and M respectively.

Remark 1. It is easy to see that if such f exists, then it is necessarily an isomorphism.

We denote by $\mathrm{YExt}(M, N)$ the set of Yoneda equivalence classes of extensions of N by M.

Example. Take $R = \mathbb{Z}$ and $N, M = \mathbb{Z}/3$. Then, an extension of $\mathbb{Z}/3$ by itself is a short exact sequence of abelian groups

$$0 \to \mathbb{Z}/3 \to E \to \mathbb{Z}/3 \to 0.$$

Hence, we have only two possibilities for E, namely, $E \cong \mathbb{Z}/9$ and $E \cong \mathbb{Z}/3 \times \mathbb{Z}/3$. On the other hand, it is easy to check that $\mathrm{YExt}(\mathbb{Z}/3, \mathbb{Z}/3)$ consists of three extensions:

- The split extension $0 \to \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3 \to \mathbb{Z}/3 \to 0$;
- The extension $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \xrightarrow{\pi} \mathbb{Z}/3 \to 0$, where the first map sends $[1] \mapsto [3]$ and the last map is the natural quotient map;
- The extension $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \xrightarrow{-\pi} \mathbb{Z}/3 \to 0$ where the first map is still $[1] \mapsto [3]$.

Our aim now is to show the following:

Theorem 3. *There is a natural bijection* $\Theta\colon\ \mathrm{YExt}(M, N) \xrightarrow{\sim} \mathrm{Ext}^1(M, N)$.

Remark 2.
- The set $\mathrm{YExt}(M, N)$ makes sense in any abelian category. On the other hand, in order to define Ext^i in an abelian category, we need that every object admits a projective resolution.
- Yoneda also constructed $\mathrm{YExt}^i(M, N)$ for $i > 0$ in any abelian category, and one also has $\mathrm{YExt}^i(M, N) \cong \mathrm{Ext}^i(M, N)$ when this latter is defined.
- There is a natural operation on $\mathrm{YExt}(M, N)$ (the Baer sum) which turns this set into a module (the split extension being the identity element). Then, the bijection above upgrades to be a module isomorphism.

The proof of the theorem will not be part of the exam, so I will only sketch it here. Feel free to fill the missing details if the material interests you.

Definition 4. An extension $\mathcal{E}\colon 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ is split if there is a map $p\colon E \to N$ such that $p \circ f = \mathrm{Id}_N$.

Remark 3.

- Equivalently, show that $\mathcal{E}$ is split if and only if there is a section $s\colon M \to E$ such that $g \circ s = \mathrm{Id}_M$. This is true in every abelian category. Is it true in the case of extensions of non-abelian groups?
- It is easy to see that any two split extensions of N by M are Yoneda equivalent (show this).

The first connection between the Ext functors and extensions come from the following:

Lemma 5. *Suppose that* $\mathrm{Ext}^1(M,N) = 0$*. Then, every extension of* N *by* M *is split.*

Proof. Consider an extension $0 \to N \to E \to M \to 0$ and apply $\mathrm{Hom}(-,N)$ to it. We obtain an extact sequence

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(E,N) \to \mathrm{Hom}(N,N) \to \mathrm{Ext}^1(M,N) = 0.$$

In particular, the map $\mathrm{Hom}(E,N) \to \mathrm{Hom}(N,N)$ is surjective. Hence, there is $p \in \mathrm{Hom}(E,N)$ which is mapped to $\mathrm{Id}_N \in \mathrm{Hom}(N,N)$. But then p clearly defines a splitting as in Definition 4. $\square$

In fact, the proof above suggests how to construct the map Θ. Pick any extension $\mathcal{E}\colon 0 \to N \to E \to M \to 0$ and apply $\mathrm{Hom}(-,N)$ to it. This yields a map

$$\cdots \to \mathrm{Hom}(N,N) \xrightarrow{\delta} \mathrm{Ext}^1(M,N) \to \cdots$$

and one defines $\Theta(\mathcal{E}) \coloneqq \delta(\mathrm{Id}_N)$. If $\mathcal{E}$ and $\mathcal{E}'$ are Yoneda equivalent, then we have a diagram

$$\begin{tikzcd}
\mathcal{E}\colon & 0 \arrow[r] & N \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="'] & 0 \\
\mathcal{E}'\colon & 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

applying $\mathrm{Hom}(-,N)$ to it, we get a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(N,N) \arrow[r] \arrow[d, "="'] & \mathrm{Ext}^1(M,N) \arrow[r] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(N,N) \arrow[r] & \mathrm{Ext}^1(M,N) \arrow[r] & \cdots
\end{tikzcd}$$

where both vertical maps are the identity. It follows that $\Theta(\mathcal{E}) = \Theta(\mathcal{E}')$, and hence that the map is well-defined. We show now how to construct the inverse $\Psi\colon\ \mathrm{Ext}^1(M,N) \to \mathrm{YExt}(M,N)$. Pick a short exact sequence

$$0 \to K \xrightarrow{f} P \xrightarrow{g} M \to 0$$

where P is projective. Apply $\mathrm{Hom}(-,N)$ to it and obtain

$$\cdots \to \mathrm{Hom}(K,N) \xrightarrow{\delta} \mathrm{Ext}^1(M,N) \to \mathrm{Ext}^1(P,N) = 0,$$

where the last vanishing is due to the fact that P is projective. Hence we get a surjection $\mathrm{Hom}(K, N) \twoheadrightarrow \mathrm{Ext}^1(M, N)$. Given an element $x \in \mathrm{Ext}^1(M, N)$, we wish to find an extension $\mathcal{E}$ such that $\Theta(\mathcal{E}) = x$. Since the map above is surjective, we can find $\beta \in \mathrm{Hom}(K, N)$ such that $\delta(\beta) = x$.
Now, suppose we can find an extension $\mathcal{E}\colon 0 \to N \to E \to M \to 0$ which fits in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\beta"] & P \arrow[r, "g"] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
$$

I claim that $\Theta(\mathcal{E}) = x$. To show this, simply apply $\mathrm{Hom}(-, N)$ to the diagram above and obtain:

$$
\begin{tikzcd}
& \beta \arrow[rr, mapsto] & & x \\
\beta \in & \mathrm{Hom}(K, N) \arrow[r, "\delta"] & \mathrm{Ext}^1(M, N) & \\
\mathrm{id}_N \in \arrow[u, mapsto] & \mathrm{Hom}(N, N) \arrow[r] \arrow[u] & \mathrm{Ext}^1(M, N) \arrow[u, "="] &
\end{tikzcd}
$$

. In order to construct E, one defines

$$
\begin{aligned}
E &\coloneqq \mathrm{coker}(K \to P \oplus N) \\
&\qquad\qquad k \mapsto (f(k), -\beta(k)).
\end{aligned}
$$

I leave it to you to check the following:

- There is a natural surjection $E \to M$ which sends the class of (p, n) to $g(p)$ and whose kernel is naturally identified with N. Thus E defines an extension of N by M.
- Another choice of β defines Yoneda equivalent extensions.

In any case, all the details can be found e.g. in Weibel's book.