Lecture 6

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1 The long exact sequence in (co)homology

The main scope of this lecture is to show how to construct the long exact sequence for Ext modules. This will follow formally by some classical propositions in homological algebra, like the snake lemma and the horseshoe lemma. As usual, we fix a ring R and we work in the category of R-modules, but once again the results are completely general.

Definition 1. A short exact sequence of complexes is given by two morphisms of chain complexes

$$0 \to A_\bullet \xrightarrow{\phi_\bullet} B_\bullet \xrightarrow{\psi_\bullet} C_\bullet \to 0$$

such that, for every $n \in \mathbb{Z}$ the induced sequence

$$0 \to A_n \xrightarrow{\phi_n} B_n \xrightarrow{\psi_n} C_n \to 0$$

is exact.

Our aim is to prove the following completely general result:

Theorem 2. *A short exact sequence of complexes*

$$0 \to A_\bullet \xrightarrow{\phi_\bullet} B_\bullet \xrightarrow{\psi_\bullet} C_\bullet \to 0$$

induces a long exact sequence on homology groups

$$\cdots H_{n+1}(C_\bullet) \xrightarrow{\delta_n} H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \xrightarrow{\delta_{n-1}} H_{n-1}(A_\bullet) \cdots$$

Here, the two middle maps are induced by the maps of the chain complexes, and the various δ_n are called the *connecting homomorphisms*, which we will have to define. By reversing indeces, one also obtains a similar result for cochain complexes. The Theorem will follow from an application of the following:

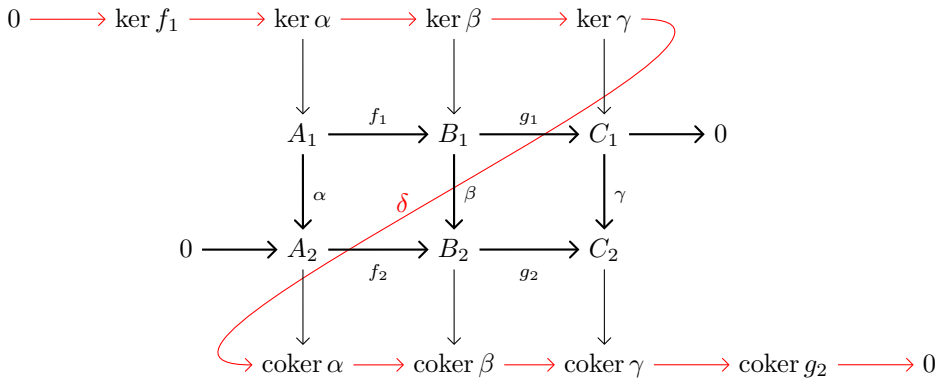
Theorem 3 (Snake lemma)**.** *A commutative diagram of exact sequences*

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & & \end{array}$$

induces a long exact sequence

$$0 \to \ker(f_1) \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma) \to \operatorname{coker}(g_2) \to 0 \tag{1.1}$$

Recall that $\operatorname{coker}(A \to B) = B/\operatorname{Im}(A)$. The name comes from the following visualization of the statement:



Remark 1. The maps between kernels and cokernels are clearly induced by the maps in the original diagram via commutativity. The only difficult part is to define the connecting homomorphism δ and to show exactness in the middle.

Proof. The proof begins with the definition of the map δ. Pick an element $c \in \ker(\gamma)$, which we visualize simply as follows:

$$\begin{array}{c} c \\ \downarrow{\scriptstyle \gamma} \\ 0 \end{array}$$

since g_1 is surjective (by the exactness assumption) we can find $b \in B_1$ such that $g_1(b) = c$:

$$\begin{array}{ccc} b & \xmapsto{g_1} & c \\ & & \downarrow{\scriptstyle \gamma} \\ & & 0 \end{array}$$

by commutativity, $\beta(b) \in \ker(g_2)$:

$$\begin{array}{ccc} b & \xmapsto{g_1} & c \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \beta(b) & \xmapsto{g_2} & 0 \end{array}$$

and finally, since f_2 is injective, there is a unique $a_2 \in A_2$ such that $f_2(a_2) = \beta(b)$:

$$\begin{array}{ccccc} & & b & \xmapsto{g_1} & c \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ a_2 & \xmapsto{f_2} & \beta(b) & \xmapsto{g_2} & 0 \end{array} \tag{1.2}$$

and we define

$$\delta(c) \coloneqq [a_2] \in A_2/\alpha(A_1)$$

(where $[\bullet]$ denotes the equivalence class in the quotient module). We begin by showing that this is well-defined, i.e., that it does not depend on any choice. The only choice that we made to define $\delta(c)$ was to choose a preimage b of c under g_1. But any other preimage of c must be of the form $b + f_1(a_1)$ for some $a_1 \in A_1$. But then

$$\beta(b + f_1(a_1)) = \beta(b) + \beta(f_1(a_1))$$

now we use that $\beta \circ f_1 = f_2 \circ \alpha$ and write

$$\beta(b) + \beta(f_1(a_1)) = \beta(b) + f_2(\alpha(a_1))$$

and so we deduce that the unique pre-image of $\beta(b + f_1(a_1))$ under f_2 is $a_2 + \alpha(a_1)$. But clearly

$$[a_2] = [a_2 + \alpha(a_1)] \in A_2/\alpha(A_1)$$

which proves the statement. Now, we prove the exactness of the sequence in the middle:

consider the portion

$$\ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta)$$

we denote, by a little abuse of notation, the map $\ker(\beta) \to \ker(\gamma)$ still by g_1 and the map $\operatorname{coker}(\alpha) \to \operatorname{coker}(\beta)$ still by f_2.

- $\operatorname{Im}(\ker(\beta)) \subset \ker(\delta)$: if $c \in \operatorname{Im}(\ker(\beta))$ then in the diagram (1.2) we can pick $b \in \ker(\beta)$, so that $\beta(b) = 0$ and then clearly $\delta(c) = 0$.
- $\ker(\delta) \subset \operatorname{Im}(\ker(\beta))$. Assume now that $\delta(c) = 0$. This happens precisely when $a_2 \in \alpha(A_1)$ in diagram (1.2). We want to show that we can pick a preimage b of c (still in the same diagram) such that $b \in \ker(\beta)$. Now, by assumption, we have $a_2 = \alpha(a_1)$. Since we are free to modify b by any element coming from A_1, we can choose $b - f_1(a_1)$ as a preimage of c. But then we compute

  $$\beta(b - f_1(a_1)) = \beta(b) - \beta(f_1(a_1)) = \beta(b) - f_2(\alpha(a_1)) = 0$$

  because $\alpha(a_1) = a_2$ by assumption and $f_2(a_2) = \beta(b)$ by construction.

This shows that $\ker(\delta) = \mathrm{Im}(\ker(\beta))$. We now show exactness at the next step:

- Suppose that $[a_2] \in \mathrm{coker}(\alpha)$ is such that $f_2[a_2] = 0 \in \mathrm{coker}(\beta)$. This simply means that $f_2(a_2) \in \beta(B_1)$, i.e., that there is $b_1 \in B_1$ such that $f_2(a_2) = \beta(b_1)$. But then I claim that $c \coloneqq g_1(b_1)$ must belong to $\ker(\gamma)$. To see this, we compute

$$\gamma(c) = \gamma(g_1(b_1)) = g_2(\beta(b_1)) = g_2(f_2(a_2)) = 0$$

  due to exactness. Hence, by reversing the construction, we see that $\delta(c) = [a_2]$.

- Finally, we show that $f_2(\delta(c)) = 0$. But this is clear by looking again at the diagram (1.2), since

$$f_2(\delta(c)) = [f_2(a_2)] = [\beta(b)] = 0 \in B_2/\beta(B_1).$$

Finally, exactness at the other steps of (1.1) is easy and left as an homework. □

We now show how the proof of Theorem 2 is a formal consequence of the snake lemma:

Proof of Theorem 2. Denote the chain complexes $A_\bullet$, $B_\bullet$ and $C_\bullet$ as

$$\cdots A_{n+1} \xrightarrow{d^A_{n+1}} A_n \xrightarrow{d^A_n} A_{n-1} \cdots,$$

$$\cdots B_{n+1} \xrightarrow{d^B_{n+1}} B_n \xrightarrow{d^B_n} B_{n-1} \cdots$$

and

$$\cdots C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \cdots$$

We apply the snake lemma two times. The first time, we consider the morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{\phi_{n+1}} & B_{n+1} & \xrightarrow{\psi_{n+1}} & C_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^A_{n+1}} & & \downarrow{\scriptstyle d^B_{n+1}} & & \downarrow{\scriptstyle d^C_{n+1}} & & \\ 0 & \longrightarrow & A_n & \xrightarrow[\phi_n]{} & B_n & \xrightarrow[\psi_n]{} & C_n & \longrightarrow & 0 \end{array}$$

then, a consequence of the snake lemma is that we get two exact sequences for any $n \in \mathbb{Z}$:

(a) $0 \to \ker(d^A_{n+1}) \to \ker(d^B_{n+1}) \to \ker(d^C_{n+1})$ and

(b) $\mathrm{coker}(d^A_{n+1}) \to \mathrm{coker}(d^B_{n+1}) \to \mathrm{coker}(d^C_{n+1}) \to 0$.

Next, we note that since $d_n \circ d_{n+1} = 0$ for each of the three complexes, we have that d_n induces a natural map

$$A_n/\,\mathrm{Im}(d^A_{n+1}) = \mathrm{coker}(d^A_{n+1}) \xrightarrow{d^A_n} \ker(d^A_{n-1})$$

whose kernel is $\ker(d_n)/\operatorname{Im}(d^A_{n+1}) = H_n(A_\bullet)$ and whose cokernel is $H_{n-1}(A_\bullet)$ (similarly for $B_\bullet$ and $C_\bullet$). Now we apply the snake lemma again to the diagram

$$\begin{array}{ccccccc} & & A_n/\mathrm{im}(d^A_{n+1}) & \longrightarrow & B_n/\mathrm{im}(d^B_{n+1}) & \longrightarrow & C_n/\mathrm{im}(d^C_{n+1}) \longrightarrow 0 \\ & & \downarrow{\scriptstyle d^A_n} & & \downarrow{\scriptstyle d^B_n} & & \downarrow{\scriptstyle d^C_n} \\ 0 & \longrightarrow & \ker(d^A_{n-1}) & \longrightarrow & \ker(d^B_{n-1}) & \longrightarrow & \ker(d^C_{n-1}) \end{array}$$

and by what we have just said, we obtain an exact sequence for every $n \in \mathbb{Z}$:

$$H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet).$$

Finally, since this is true for every n, one easily checks that all these sequences 'glue' to yield the long exact sequence in the statement. □

1.1 Application: long exact sequence of Ext modules

We wish to use the results above to construct the long exact sequence of Ext modules. As usual, we fix a ring R and a R-module N, and we consider the contravariant functor $A \mapsto F(A) = \mathrm{Hom}(A, N)$. Recall that given a short exact sequence

$$0 \to A \to B \to C \to 0$$

we wish to show that there is a natural long exact sequence

$$0 \to F(C) \to F(B) \to F(A) \to \mathrm{Ext}^1(C, N) \to \mathrm{Ext}^1(B, N) \to \mathrm{Ext}^1(A, N) \to \mathrm{Ext}^2(C, N) \to \cdots$$

To do this, we would like of course to use Theorem 2. But there is one last missing step: given a projective resolution $P_\bullet \to A$ and a projective resolution $R_\bullet \to C$ we need to construct a third projective resolution $Q_\bullet \to B$ which sits in the following commutative diagram (all rows are exact)

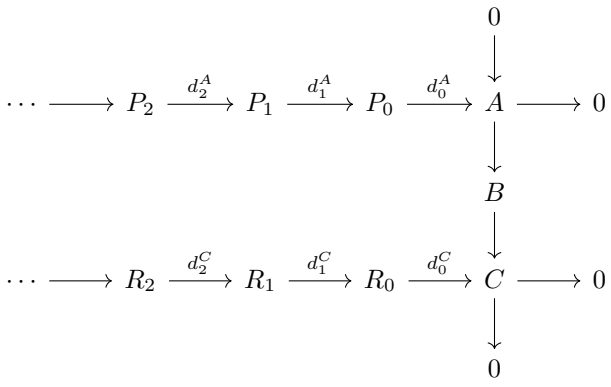
$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_\bullet & \longrightarrow & Q_\bullet & \longrightarrow & R_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Suppose that we can construct such $Q_\bullet$. Then we can apply F to the short exact sequence $0 \to P_n \to Q_n \to R_n \to 0$ and using that R_n is projective, we get a short exact sequence $0 \to F(R_n) \to F(Q_n) \to F(P_n) \to 0$ and hence a short exact sequence of cochain complexes

$$0 \to F(R_\bullet) \to F(Q_\bullet) \to F(P_\bullet) \to 0$$

which will yield the long exact sequence in Ext groups (by using the dual version of Theorem 2 for cochain complexes). The procedure to construct $Q_\bullet$ is called the horse-

shoe lemma. Once again, the name comes from the picture



We shall show in the next lecture that we can put $Q_n = P_n \oplus R_n$. The only hard part will be to construct the maps d_n^B. Note that we cannot simply take $d_n^B = d_n^A \oplus d_n^C$ (for instance, this does not make sense in general for $n = 0$). This is to say that altough at each degree Q_n is the direct sum of P_n and R_n, the whole complex $Q_\bullet$ is *not* in general the direct sum $P_\bullet$ and $R_\bullet$. On the other hand, if $B = A \oplus C$, then one easily checks that $Q_\bullet = P_\bullet \oplus R_\bullet$ makes sense and is a projective resolution of B. What consequence does this have on the long exact sequece of Ext groups, in this case?