Lecture 5

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1 The Ext modules are well-defined

The aim of this lecture is to show that the Ext modules, as defined in the previous lecture, are actually well-defined (do not depend on any choice).

We denote a chain complex of R-modules by $(F_\bullet, f_\bullet)$, so

$$(F_\bullet, f_\bullet) = \cdots F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \cdots$$

Definition 1. Let $(F_\bullet, f_\bullet)$ and $(G_\bullet, g_\bullet)$ be chain complexes. A morphism $\psi_\bullet \colon (F_\bullet, f_\bullet) \to (G_\bullet, g_\bullet)$ is given by a collection of (R-module) morphisms $\psi_i \colon F_i \to G_i$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "f_{n+2}"] & F_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\psi_{n+1}"] & F_n \arrow[r, "f_n"] \arrow[d, "\psi_n"] & F_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\psi_{n-1}"] & \cdots \\
\cdots \arrow[r, "g_{n+2}"] & G_{n+1} \arrow[r, "g_{n+1}"] & G_n \arrow[r, "g_n"] & G_{n-1} \arrow[r, "g_{n-1}"] & \cdots
\end{tikzcd}$$

meaning that $\psi_n \circ f_{n+1} = g_{n+1} \circ \psi_{n+1}$ for every $n \in \mathbb{Z}$.

Remark 1. The set of morphisms between two chain complexes form naturally a R-module.

Proposition 2. *Any morphism of chain complexes $\psi_\bullet \colon (F_\bullet, f_\bullet) \to (G_\bullet, g_\bullet)$ induces a morphism on the homology groups.*

Proof. In order to prove the proposition we only need to show that for every $i \in \mathbb{Z}$ we have that (a) ψ_i sends $\ker(f_i)$ to $\ker(g_i)$ and (b) that $\mathrm{Im}(f_{i+1})$ to $\mathrm{Im}(g_{i+1})$.

(a) If $x \in \ker(f_i)$ then $0 = \psi_{i-1}(f_i(x)) = g_i(\psi_i(x))$.

(b) If $x \in \mathrm{Im}(f_{i+1})$ then there is some $y \in F_{i+1}$ such that $f_{i+1}(y) = x$. Then we compute $\psi_i(x) = \psi_i(f_{i+1}(y)) = g_{i+1}(\psi_{i+1}(y)) \in \mathrm{Im}(g_{i+1})$.

Recall that the homology groups are R-modules. Verify that the induced maps are also a R-module morphisms. $\square$

If $\psi = \psi_\bullet$ is a morphism of chain complexes, we denote by $H_n(\psi)$ the morphism induced on the n-th homology group. In general, the category of chain complexes over a ring R is, for many reasons, too big for our considerations, meaning that there are many different chain complexes that we would like to consider equivalent (this is in analogy with topology, where you may want to consider to spaces equivalent if they are homotopic). For instance, we may only be interested in the homology of a cochain complex, so we can call two chain complexes equivalent if there are two morphisms $\psi\colon F_\bullet \to G_\bullet$ and $\phi\colon G_\bullet \to F_\bullet$ such that induced morphism on each homology group are one the inverse of the other. In the end of this lecture, we shall define a finer equivalence relation of this one: homotopy.

1.1 Projective modules

Projective modules are a generalization of free modules. The fact is that the definition of a projective module can be made in terms of arrows and can therefore be applied in any category.

Definition 3. We say that a R-module P is projective if for any surjection of R-modules $\pi\colon A \twoheadrightarrow B$ and any map $P \xrightarrow{f} B$ we can find a dotted arrow (a "lift")

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "f"] & B. \end{tikzcd}$$

which makes the diagram above commutative.

Note that the dotted arrow is in general not unique, as the next example shows.

Proposition 4. *Any free module is projective.*

Proof. We need to check that we can always fill the dotted arrow above when $P = R^I$ is a free module. Let e_i with $i \in I$ be the canonical basis of R^I. Note that by the universal property of direct sum, a morphism from R^I to another module A is determined by prescribing an element $a_i \in A$ for every $i \in I$. Then, the associated map $R^I \to A$ is the unique one that sends e_i to a_i. Write $b_i \coloneqq f(e_i)$. Since π is surjective, we can find $a_i \in A$ such that $\pi(a_i) = b_i$ for every $i \in I$. We can then define the lift as the map $R^I \to A$ which sends e_i to a_i. □

The motivation behing the definition of projective module comes from the following observation: a module P is projective if and only if the functor $\mathrm{Hom}(P, -)$ is exact.

Remark 2. In general, the covariant functor $\mathrm{Hom}(P, -)$ is only left-exact. Asking for it to be exact is the same as asking that for any surjection $\pi\colon A \twoheadrightarrow B$ the induced map $\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B)$ is surjective. This is clearly equivalent to the definition of projective module.

It is easy to characterize projective modules:

Proposition 5. *A module P is projective if and only if there is another module M such that $P \oplus M$ is free.*

Proof. We prove both implication. So assume that P is projective, we need to exhibit P as a direct summand of a free module. We choose a free module F together with a surjection $\pi\colon F \to P$ (we can always do this, see the previous lecture notes). Now, we use that P is projective and consider the following diagram

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\mathrm{Id}_P"] & P. \end{tikzcd}$$

and we choose a dotted arrow that makes it commutative, say we call if $h\colon P \to F$. Note that h must be necessarily injective (why?). Now, we consider $\Pi := h \circ \pi$ which is an endomorphism of F.

Claim 5.1. The map Π is a projector, i.e., $\Pi \circ \Pi = \Pi$.

Proof of claim. In fact, we compute $\Pi \circ \Pi = h \circ \pi \circ h \circ \pi$ and we notice that $\pi \circ h = \mathrm{Id}_P$.

In general, everytime one has a projector $\Pi\colon M \to M$ on a module M, one can decompose $M = \mathrm{Im}(P) \oplus \ker(\Pi)$ (homework!). So, we can write $F = \ker(\Pi) \oplus \mathrm{Im}(\Pi)$. But $\mathrm{Im}(\Pi) = h(\pi(F)) = h(P)$ because π is surjective. Since h is injective, we can identify $P \cong h(P)$.

To prove the other direction, we assume that there is a module M such that $P \oplus M = F$ is free, and we need to show that P is projective. So we consider any diagram

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "f"] & B. \end{tikzcd}$$

and we need to find a dotted arrow. Now, we consider another diagram obtained from the previous one as follows

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ P \oplus M \arrow[ur, dashed, "\exists"] \arrow[r, "\tilde{f}"] & B. \end{tikzcd}$$

where we define $\tilde{f}(p, m) := f(p)$ for every $(p, m) \in P \oplus M$. Since F is free, it is projective, so we can fill the dotted arrow with some $\tilde{h}\colon P \oplus M \to A$. Then, the composition

$$h\colon P \hookrightarrow P \oplus M \xrightarrow{\tilde{h}} A$$

defines the required lifting, where the first map is simply the inclusion of P in $P \oplus M$ given by $p \mapsto (p, 0)$. $\square$

As I said before, in general, projective resolutions will be used in lieu of free resolutions.

Definition 6. A projective resolution of a module A is by definition a resolution (= an exact chain complex)

$$\cdots P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

where each P_i is projective for $i \geq 0$.

One of the main properties of projective resolutions (which holds in general abelian categories) is the following important fact:

Theorem 7. *Assume that $(P_\bullet, f_\bullet)$ is a projective resolution of A and that $(Q_\bullet, g_\bullet)$ is a projective resolution of B. Then, for any module morphism $\psi\colon A \to B$ we can find a morphism of chain complexes $\psi_\bullet\colon P_\bullet \to Q_\bullet$ lifting ψ, i.e., such that the following commutes*

$$\begin{tikzcd}
\cdots \arrow[r, "f_3"] & P_2 \arrow[r, "f_2"] \arrow[d, "\psi_2"] & P_1 \arrow[r, "f_1"] \arrow[d, "\psi_1"] & P_0 \arrow[r, "f_0"] \arrow[d, "\psi_0"] & A \arrow[r] \arrow[d, "\psi"] & 0 \\
\cdots \arrow[r, "g_3"] & Q_2 \arrow[r, "g_2"] & Q_1 \arrow[r, "g_1"] & Q_0 \arrow[r, "g_0"] & B \arrow[r] & 0.
\end{tikzcd}$$

Proof. We prove this by induction. We put for simplicity $P_{-1} = A$, $Q_{-1} = B$ and $\psi_{-1} = \psi$. The first step is to show that we can find ψ_0. So, we need to find a dotted arrow

$$\begin{tikzcd}
P_0 \arrow[r, "f_0"] \arrow[d, dotted] & A \arrow[r] \arrow[d, "\psi"] & 0 \\
Q_0 \arrow[r, "g_0"] & B \arrow[r] & 0
\end{tikzcd}$$

making the diagram commutative. We use that P_0 is projective and that g_0 is surjective, to find a map $\psi_0\colon P_0 \to Q_0$ which lifts the map $\psi \circ f_0\colon P_0 \to B$. Now, assume that we have found ψ_i for every $i \geq n$ and we want to construct ψ_{n+1}. We consider the diagram

$$\begin{tikzcd}
P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, dotted] & P_n \arrow[r, "f_n"] \arrow[d, "\psi_n"] & P_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\psi_{n-1}"] & \cdots \\
Q_{n+1} \arrow[r, "g_{n+1}"] & Q_n \arrow[r, "g_n"] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

since both complexes are exact, we have that $\ker(g_n) = \mathrm{Im}(g_{n+1})$. Hence, in order to use the projectivity of P_{n+1}, we only have to show that the composition

$$\begin{tikzcd}
P_{n+1} \arrow[r, "f_{n+1}"] & P_n \arrow[d, "\psi_n"] \\
& Q_n
\end{tikzcd}$$

lands in $\ker(g_n) = \operatorname{Im}(g_{n+1})$. If this is the case, we can apply the same reasoning as for the zeroth step. So, we only need to show that $g_n \circ \psi_n \circ f_{n+1} = 0$. We compute

$$g_n \circ \psi_n \circ f_{n+1} = \psi_{n-1} \circ f_n \circ f_{n+1} = 0$$

because $g_n \circ \psi_n = \psi_{n-1} \circ f_n$ by induction and $f_n \circ f_{n+1} = 0$ because $F_\bullet$ is a complex. $\square$

In general, the map of chain complexes lifting ψ is not unique. But we can understand how two different lifting are related to each-other. To do this, let us begin from the first step in the previous proof. If ψ_0' is another lifting, then we clearly have that $\psi_0 - \psi_0'$ has image in $\ker(g_0)$, since

$$g_0 \circ (\psi_0 - \psi_0') = g_0 \circ \psi_0 - g_0 \circ \psi_0' = \psi \circ f_0 - \psi \circ f_0 = 0.$$

Since $\ker(g_0) = \operatorname{Im}(g_1)$ by exactness, we can use the projectivity of P_0 to lift the map $\psi_0 - \psi_0' \colon P_0 \to \operatorname{Im}(g_1)$ to a map $h_0 \colon P_0 \to P_1$. **Upshot**: there is a map $h_0 \colon P_0 \to Q_1$ such that $\psi_0 - \psi_0' = g_1 \circ h_0$.

Now, we look at the next degree, and we want to understand the difference $\psi_1 - \psi_1'$. But this time the morphism $\psi_1 - \psi_1'$ does not necessarily land in $\operatorname{Im}(g_2)$. Nevertheless, it would if $\psi_0 = \psi_0'$, as it happened before. This suggests that the 'obstruction' to repeat the procedure above and find $h_1 \colon P_1 \to Q_2$ is given by the difference $\psi_0 - \psi_0'$. To make this formal, we apply g_1 and compute

$$g_1 \circ (\psi_1 - \psi_1') = \psi_0 \circ f_1 - \psi_0' \circ f_1 = g_1 \circ h_0 \circ f_1.$$

From the equation above we gather that

$$g_1 \circ (\psi_1 - \psi_1' - h_0 \circ f_1) = 0$$

and hence that $\psi_1 - \psi_1' - h_0 \circ f_1$ lands in $\ker(g_1) = \operatorname{Im}(g_2)$. So we can again use that P_1 is projective and find a map $h_1 \colon P_1 \to Q_2$ such that $\psi_1 - \psi_1' = g_2 \circ h_1 + h_0 \circ f_1$. But we can repeat this procedure indefinitively and find that

$$\boxed{\text{There are maps } h_i \colon P_i \to Q_{i+1} \text{ for } i \geq 0 \text{ such that } \psi_i - \psi_i' = g_{i+1} \circ h_i + h_{i-1} \circ f_i.}$$

Definition 8. Two morphisms $\psi, \psi' \colon F_\bullet \to G_\bullet$ of chain complexes are homotopy equivalent if there are maps $h_\bullet \colon P_\bullet \to Q_{\bullet+1}$ which satisfies the relations boxed above.

Remark 3. There is an equivalent notion for cochain complexes, it is just a matter of reorganizing the indeces.

The main fact is the following:

Proposition 9. *Two homotopic morphisms induce the same map in homology.*

Proof. In fact, pick any $\alpha \in H_i(F_\bullet)$, and choose a representative $a \in \ker(f_i)$. Now, we compute

$$\psi_i(a) - \psi_i'(a) = (g_{i+1} \circ h_i)(a) + (h_{i-1} \circ f_i)(a) \overset{a \in \ker(f_n)}{=} (g_{i+1} \circ h_i)(a)$$

and hence $\psi_i(a) - \psi_i'(a) \in \operatorname{Im}(g_{i+1})$. But the $\psi_i(a) = \psi_i'(a)$ when seen as elements of $H_i(G_\bullet) = \ker(g_i)/\operatorname{Im}(g_{i+1})$. $\square$

Having defined homotopy equivalence of morphisms, it is easy to extend it to chain complexes

Definition 10. We say that two chain complexes $F_\bullet$ and $G_\bullet$ are homotopic if there are two morphisms $\psi\colon F_\bullet \to G_\bullet$ and $\phi\colon G_\bullet \to F_\bullet$ such that $\psi \circ \phi$ is homotopy equivalent to the identity of $G_\bullet$ and $\phi \circ \psi$ is homotopy equivalent to the identity of $F_\bullet$.

Now, what we have proved before can be summarized as follows: let $P_\bullet$ be a projective resolution of A and let $Q_\bullet$ be a projective resolution of B and let $\psi\colon A \to B$ be a morphism. Then any two lifts $\psi_\bullet, \psi'_\bullet\colon P_\bullet \to Q_\bullet$ are homotopic equivalent.

Corollary 11. *If $\psi_\bullet\colon P_\bullet \to P_\bullet$ is a lift of the identity $\mathrm{Id}_A\colon A \to A$, then $\psi_\bullet$ is homotopic to the identity of $A_\bullet$.*

Proof. In fact, both the identity of $A_\bullet$ and $\psi_\bullet$ are lifts of the identity of A. $\square$

This leads to the proof of our main result, namely, that Ext modules are well-defined. We recall that we fix a module N and we consider the contravariant functor $F = \mathrm{Hom}(-, N)$. To define the Ext modules $\mathrm{Ext}^i(A, N)$ we choose a projective resolution $P_\bullet$ of A and apply F to it and obtain a cochain complex

$$F(P_\bullet) = \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \cdots .$$

Then we define $\mathrm{Ext}^i(A, N) = H^n(F(P_\bullet))$. We prove

Theorem 12. *Let $P_\bullet, Q_\bullet$ be two projective resolutions of A. Then for any n we have a natural isomorphism $H_n(F(P_\bullet)) \cong H_n(F(Q_\bullet))$.*

Proof. Pick any lift $\psi\colon P_\bullet \to Q_\bullet$ and $\phi\colon P_\bullet \to Q_\bullet$ of the identity of A (considered in both directions). We know that $\phi \circ \psi$ is homotopic to the identity of A via some homotopy $h_\bullet$ because of the corollary above. Now, we apply F to obtain maps $F(\psi)\colon F(Q_\bullet) \to F(P_\bullet)$ and $F(\phi)\colon F(P_\bullet) \to F(Q_\bullet)$ (recall that F is contravariant). But then also $F(\psi) \circ F(\phi)$ is homotopic to the identity of $F(P_\bullet)$ via the maps $F(h_\bullet)$ (why?). Hence, the induced maps in cohomology

$$H^n(F(\phi))\colon H^n(F(P_\bullet)) \to H^n(F(Q_\bullet))$$

and

$$H^n(F(\psi))\colon H^n(F(Q_\bullet)) \to H^n(F(P_\bullet))$$

are one the inverse of the other. Finally, the map $F(\phi)$ does not depend on the chosen lift ϕ, because we know that any other lift is homotopic to ϕ. $\square$