

Lecture 4

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1 First notions of homological algebra

Let R be a ring and fix a module M over R . In this lecture we consider the association

$$A \mapsto \operatorname{Hom}(A, M) =: F(A)$$

where A is any R -module. Note that then also $F(A)$ is a module, where addition and scalar multiplication are inherited from the target. For instance, if $M = R$, then $F(A) = A^\vee = \operatorname{Hom}(A, R)$ is by definition the dual of A . When R is a field, this corresponds to the usual dual of vector spaces.

The considerations that we are going to make today and in the following lectures belong to the subject of *homological algebra*. This is a broad and general subject whose techniques are applied to many different areas of mathematics, one of them being modules over a ring. The constructions and considerations that we are going to make are completely general, and work in basically every abelian category which appears in nature (for instance, categories of sheaves in geometry, categories of Galois modules in number theory).

Lemma 1. Any map of R -modules $g: A \rightarrow B$ induces a map $F(g): F(B) \rightarrow F(A)$.

Proof. The map $F(B) \rightarrow F(A)$ is given by associating to any $g \in F(B) = \operatorname{Hom}(B, M)$ the composition $g \circ f \in \operatorname{Hom}(A, M)$. One easily sees that this is a map of R -modules. $\square$

Note also this also behaves well with composition, that is, if we have two maps $g: A \rightarrow B$ and $h: B \rightarrow C$ then $F(h \circ g) = F(g) \circ F(h)$. In this sense, we say that F is a contravariant *functor*. A functor is a morphism between categories. I do not want to say what is a category, but I can say what a functor on R -modules is:

Definition 2. A covariant functor on R -modules is a law F which associated to any module A another module $F(A)$ and for any map $g: A \rightarrow B$ a map $F(g): F(A) \rightarrow F(B)$. This association must also behave well under composition. Similarly, we say that the functor is contravariant if it reverses the order of the arrows.

For instance, we can also consider the functor $A \mapsto \operatorname{Hom}(M, A)$. This is now covariant.

The first question is: how does F behave with respect to injective or surjective maps, and to exact sequences?

Lemma 3. *If $g: A \rightarrow B$ is surjective, then $F(g): F(B) \rightarrow F(A)$ is injective.*

Proof. Pick $u \in F(B) = \text{Hom}(B, M)$ and assume that $u \circ g = 0 \in F(A)$. This means that for every $a \in A$ $u(g(a)) = 0$. But g is surjective, hence, for any $b \in B$ we also have that $u(b) = 0$. Hence $u = 0$ and the map is injective. $\square$

Assume now that $g: A \rightarrow B$ is injective. Then, the map $F(g)$ is in general not surjective. One example can be given as follows: pick any ring R and a non-zero divisor $r \in R$. Consider the injective map of R -modules $R \xrightarrow{r} R$. Then, what is the induced map $F(R) \rightarrow F(R)$? It is simply the map that sends $u \in \text{Hom}(R, M) = M$ to $r \cdot u$. Thus, after identifying $F(R)$ with M , we see that $F(r)$ is nothing but $M \xrightarrow{r} M$ which is in general neither injective (if for instance M has r -torsion) nor surjective (think $R = \mathbb{Z}$ and $r > 1$).

Lemma 4. *Let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{h} C \rightarrow 0$ be exact. Then*

$$0 \rightarrow F(C) \xrightarrow{F(h)} F(B) \xrightarrow{F(g)} F(A)$$

is exact.

Proof. We know that the first map is injective by the previous Lemma, so we only need to show that $\ker(F(g)) = \text{Im}(F(h))$. We prove both containments. So, pick $u \in F(B)$ and assume that $u \in \text{Im}(F(h))$. This simply means that $u = v \circ h$ for $v \in F(C)$. Now, $F(g)(u) = u \circ g = v \circ h \circ g$ and $h \circ g = 0$ because the original sequence is exact. For the other containment, assume that $u \in \ker(F(g))$. Again, this means that the composition $u \circ g = 0$. Thus we obtain the commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ & \searrow 0 & \downarrow u & & \\ & & M & & \end{array}$$

which means that $\text{Im}(g) \subset \ker(u)$. Hence, u descends to a morphism $u: B/\text{Im}(g) \cong C \rightarrow M$ and hence we can complete the diagram

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ & \searrow 0 & \downarrow u & \swarrow \tilde{u} & \\ & & M & & \end{array}$$

which shows that $u = \tilde{u} \circ h$ and hence that $u \in \text{Im}(F(h))$. $\square$

In this sense, we say that F is a left-exact contravariant functor. The main goal of homological algebra is then to study systematically the failure of the surjection on the sequence above. The main construction is to associate to any module A a sequence of modules $H^i(A)$ for $i \geq 0$ such that $H^0(A) = F(A)$ and for any exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ we get a long exact sequence

$$0 \rightarrow F(C) \rightarrow F(B) \rightarrow F(A) \rightarrow H^1(C) \rightarrow H^1(B) \rightarrow H^1(A) \rightarrow H^2(C) \cdots \quad (1.1)$$

The way in which we shall construct the various H^i and the long exact sequence above has nothing to do with modules in particular, and in fact the same construction works over any abelian category (once one has a left/right exact functor plus some mild technical conditions).

In our particular case, when $F = \text{Hom}(-, M)$, the groups $H^i(A)$ are called Ext groups $H^i(A) = \text{Ext}^i(A, N)$. Our aim in these lectures is to construct these groups and to show that they satisfy the required properties.

1.1 The role of free objects

Let us show one instance when if $g: A \rightarrow B$ is injective, then we know that $F(g)$ is surjective.

Lemma 5. *Assume that $g: A \rightarrow B$ is injective and that there is a retraction $r: B \rightarrow A$, i.e., such that $r \circ g = \text{Id}$. Then $F(B) \rightarrow F(A)$ is surjective.*

Proof. We have induced maps $F(A) \xrightarrow{F(r)} F(B) \xrightarrow{F(g)} F(A)$ whose composition is $F(\text{Id}_A) = \text{Id}_{F(A)}$. But then $F(g)$ must be surjective. $\square$

A particular example is the following:

Lemma 6. *Let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{h} C \rightarrow 0$ be a short exact sequence and assume that C is free, $C \cong R^I$. Then, $F(B) \rightarrow F(A)$ is surjective and in particular the sequence*

$$0 \rightarrow F(C) \xrightarrow{F(h)} F(B) \xrightarrow{F(g)} F(A) \rightarrow 0$$

is exact.

Proof. We only need to prove surjectivity of the map $F(g)$. We want to apply the previous lemma. This follows by the following lemma: $\square$

Lemma 7. *Assume that $0 \rightarrow A \xrightarrow{g} B \xrightarrow{h} C \rightarrow 0$ is exact and that $C \cong R^I$ is free. Then, we have an isomorphism $B \cong A \oplus C$ compatible with the maps above, and in particular the map $A \rightarrow B$ has a retraction.*

Proof. Let $e_i \in R^I$ be the natural basis, for $i \in I$. Then, we know by the universal product of direct sum, that to give a map $R^I \rightarrow B$ is the same as to give an element $b_i \in B$ for any $i \in I$. The map is then the unique one that sends e_i to b_i . So, we choose $b_i \in B$ such that $h(b_i) = e_i$. This defines a map $s: R^I \rightarrow B$, which is injective because $h \circ s = \text{Id}$. We want to show that $C \cong g(A) \oplus s(C)$. But $g(A) \cap s(C) = 0$ because if $b \in B$ is an element in this intersection, then we can write $b = g(a) = s(c)$ but $h(b) = 0$ since $b \in g(A) = \ker(h)$ and also $h(b) = h(s(c)) = c$ hence $c = 0$. So $g(A) \oplus s(C) \subset B$. To prove equality, notice that for any $b \in B$ then $b - s(h(b)) \in g(A)$. $\square$

1.2 Free resolutions

To compute the Ext modules, we shall use free (later, projective) resolutions. The reason why this is the case will be clear in the later lectures.

Definition 8. A complex of modules is a sequence of maps of R -modules

$$\cdots \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \cdots$$

such that $f_i \circ f_{i+1} = 0$ for every $i \in \mathbb{Z}$. A cochain complex of modules is the same thing but with increasing indices.

Usually, a cochain complex satisfies $A_i = 0$ for $i < 0$, so most of the times it looks like

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \rightarrow 0$$

and a cochain complex like

$$0 \rightarrow A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \cdots$$

We denote a complex by $(M_\bullet, f_\bullet)$.

Definition 9. If $(M_\bullet, f_\bullet)$ is a complex, we define its homology groups as

$$H_i(M_\bullet) := \frac{\ker(f_i)}{\operatorname{Im}(f_{i+1})}.$$

In particular, a complex is exact (as a sequence of maps) if and only if $H_i(A_\bullet) = 0$ for every i .

Remark 1. If $M_\bullet$ is a chain complex, then we can apply $\operatorname{Hom}(-, M) = F$ to it, and obtain a cochain complex (because F respects composition, and sends the zero map to the zero map).

The way one computes Ext groups (or cohomology in general of contravariant factors) is using free (or projective) resolutions.

Definition 10. A free-resolution of a module A is a **exact** chain complex

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A \rightarrow 0$$

where each A_i is a **free** R -module for $i \geq 0$. So, in particular, $\operatorname{coker}(f_1) \cong A$.

Lemma 11. Any module A admits a free resolution.

Proof. Consider the free module R^A generated by elements of A . So the canonical basis of R^A is e_a for $a \in A$ and any element of R^A can be written uniquely as $\sum_{a \in A} r_a e_a$ where $r_a \in R$ and $r_a \neq 0$ only for finitely $a \in A$. There is a natural map $R^A \rightarrow A$ which sends $e_a \mapsto a$. This is clearly surjective, and the kernel $K_1 \subset R^A$ is again an R -module. Now, we apply the same construction for K_1 , and we obtain a surjection $R^{K_1} \rightarrow K_1$ with kernel K_2 . By reiterating this, we obtain a sequence of maps

$$\cdots \rightarrow R^{K_2} \rightarrow R^{K_1} \rightarrow R^A \rightarrow A \rightarrow 0$$

which is exact by construction. $\square$

Note that if A is Noetherian, then we can construct a free resolution all of whose terms are finitely generated (why?).

Remark 2. If A is a Noetherian module, it is not true in general that A admits a finite free resolution. In fact, Serre showed that every Noetherian module A admits a finite free resolution if and only if the ring R is regular (a definition from algebraic geometry). This was one of the first instances where homological algebra was used in algebraic geometry.

For instance, if $a \in R$ is not a zero-divisor, then $R \xrightarrow{a} R \rightarrow R/(a) \rightarrow 0$ is a free resolution of $R/(a)$.

Another example: let now a field K and let $R = K[x, y]$. Then we can see K as a R -module via the isomorphism $K \cong K[x, y]/(x, y)$. If we want to resolve K , then we can start with the quotient map above

$$K[x, y] \rightarrow K \rightarrow 0$$

whose kernel is the maximal ideal (x, y) . Now, consider the surjective map $R^2 \rightarrow (x, y)$ which sends e_1 to x and e_2 to y . So we obtain

$$K[x, y]^2 \rightarrow K[x, y] \rightarrow K \rightarrow 0 \quad (1.2)$$

$$e_1 \rightarrow x \quad (1.3)$$

$$e_2 \rightarrow y \quad (1.4)$$

Then, the kernel K of this last map is given by pairs $(f, g) \in R^2$: $fx + gy = 0$. Since $K[x, y]$ is a UFD and since both x and y are irreducible hence primes, x must divide g and y must divide f and then we necessarily have that the kernel is $\{(yh, -xh) : h \in R\} \subset R^2$. In particular, the map $R \rightarrow K$ sending h to $(yh, -xh)$ is an isomorphism, and we finish our resolution

$$0 \rightarrow K[x, y] \rightarrow K[x, y]^2 \rightarrow K[x, y] \rightarrow K \rightarrow 0.$$

Note that resolutions are not unique. But, as we shall see, they are always *homotopic* one to the other.

Let us now prove a useful lemma.

Lemma 12. *Let $A_\bullet$ be a free resolution of A . Let $F(A_\bullet)$ be the associated cochain complex. Then, $H^0(F(A_\bullet)) \cong F(A)$.*

This says that the kernel of $F(A_0) \rightarrow F(A_1)$ does not depend on the chosen resolution.

Proof. Consider the free resolution $\cdots A_1 \xrightarrow{f_0} A_0 \xrightarrow{\pi} A$. Then we obtain two short exact sequences

$$0 \rightarrow \ker(\pi) \rightarrow A_0 \xrightarrow{\pi} A \rightarrow 0$$

and hence that $F(A) = \ker(F(A_0) \rightarrow F(\ker(\pi)))$. Then, we obtain another sequence for $\ker(\pi)$, namely

$$0 \rightarrow \ker(f_0) \rightarrow A_1 \rightarrow \ker(\pi) \rightarrow 0$$

from which we deduce that $F(\ker(\pi)) \rightarrow F(A_1)$ is injective. Hence,

$$F(A) = \ker(F(A_0) \rightarrow F(\ker(\pi))) = \ker F(A_0) \rightarrow F(A_1).$$

□

Note that we haven't used that all the A_i are free in this lemma. One of the main results that we are going to prove in the next lectures is this:

Theorem 13. *For any free resolution $A_\bullet$ of A , the cohomology groups H^i of the cochain complex $F(A_\bullet)$ do not depend on the chosen resolution. One then defines $\text{Ext}^i(A, M) = H^i(F(A_\bullet))$.*

So $\text{Ext}^0(A, M) = F(A) = \text{Hom}(A, M)$. For instance, let us compute Ext for our examples.

1. If R is a ring and $a \in R$ is not a zero divisor, then $R \xrightarrow{a} R$ is a free resolution of $R/(a)$. Let us now compute $\text{Ext}^i(R/(a), M)$. All we have to do is to apply $\text{Hom}(-, M)$ to the free resolution and then compute cohomology. But $\text{Hom}(R, M) = M$ hence we get a cochain complex $M \xrightarrow{a} M$. Then $\text{Ext}^0(R/(a), M) = M[a]$ and $\text{Ext}^1(R/(a), M) = M/aM$.
2. If $R = K[x, y]$ and $M = R$ let us compute $\text{Ext}^i(K, R)$ where $K \cong K[x, y]/(x, y)$. We use the free resolution from above. By applying $\text{Hom}(-, R)$ we find a cochain complex

$$R \rightarrow R^2 \rightarrow R$$

where the first map sends $1 \mapsto (x, y)$ and the second map sends $(1, 0) \mapsto y$ and $(0, 1) \mapsto -x$. Thus we see that $\text{Ext}^i(K, R) = 0$ for $i = 0, 1$ and $\text{Ext}^2(K, R) = K$. All the others are zero.

The plan for the following lectures:

1. Substitute free resolution with projective resolutions.
2. Show that any two projective resolutions are homotopy equivalent, hence they yield the same cohomology groups.
3. Show that with this construction we obtain the long exact sequence mentioned in (1.1).