Lecture 2

Domenico Valloni

Definition 1. Let R be a ring, and let M be a module over it.

- We say that M is Noetherian (after E. Noether) if every ascending chain of submodules
$$M_1 \subset M_2 \subset M_3 \subset \cdots \subset M$$
stabilizes, i.e., $M_i = M_{i+1}$ for every $i >> 0$.
- We say that M is Artian (after E. Artin) if every descending chain of submodules
$$\cdots \subset M_3 \subset M_2 \subset M_1 \subset M$$
stabilizes, i.e., $M_i = M_{i+1}$ for every $i >> 0$.

We say that a ring R is Noetherian (resp. Artinian) if it is so when considered as a module over itself.

In particular, a ring is Noetherian if every ascending chain of ideals
$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset M$$
stabilizes eventually (since submodules are ideals in this case). Examples:

1. If $R = \mathbb{Z}$ then any finite abelian group is both Noetherian and Artinian (since it is finite, every chain must stabilize)
2. If $R = M = \mathbb{Z}$ then M is noetherian but not artinian. It is not artinian because
$$\cdots \subset (2)^3 \subset (2)^2 \subset (2) \subset \mathbb{Z}$$
is a non-stabilizing descending chain of ideals. Let us now show that it is noetherian. We know that every ideal is principal, so we can write an ascending chain of ideals as
$$(n_1) \subset (n_2) \subset (n_3) \subset \cdots$$
with $n_i \in \mathbb{Z}$. But then n_i must divide n_j for every $j \leq i$. Since n_1 has only finitely many divisors, we conclude that the chain must stabilize.
3. Similarly, one checks that if k is a field, then $k[x]$ is noetherian over itself, but not artinian.

4. The abelian group $\mathbb{Q}/\mathbb{Z}$ seen as a module over $\mathbb{Z}$ is neither noetherian nor artinian. To show this, consider the following chain, which infinite in both directions ($n \in \mathbb{N}$):

$$\cdots p^n\mathbb{Z} \subset p^{n-1}\mathbb{Z} \subset \cdots \subset p\mathbb{Z} \subset \mathbb{Z} \subset \frac{1}{p}\mathbb{Z} \subset \cdots \subset \frac{1}{p^n}\mathbb{Z} \subset \frac{1}{p^{n+1}}\mathbb{Z} \subset \cdots$$

5. There are, on the other hand, many Artinian modules over $k[x]$. For instance, pick $n \geq 1$ and consider $M = k[x]/(x)^n$. What are the submodules of M? By the correspondence theorem, they corresponds to ideals $(x^n) \subset I \subset k[x]$. Since $k[x]$ is a P.I.D., the ideal $I = (f)$ for some polynomial $f \in k[x]$ and the inclusion means that f divides x^n. But then the only possible choices are $I = (x)^i$ for $i = 0, 1, \cdots n$. In particular, $k[x]/(x^n)$ has only finitely many submodules, hence it is automatically both artinian and noetherian.

There are also examples of modules that are artinian and not noetherian, but those are to be considered pathological, as we shall now explain.

Remark 1. For a ring R, being Artinian is way more stringent than being Noetherian. In fact, one can show that if R is a ring which is artinian, then it is automatically noetherian (Akizuki's theorem). We shall not pursue this. Let me just say that noetherian rings appear everywhere and satisfy many useful properties, and that most of algebraic geometry has the noetherian assumption. On the other hand, artinian rings appear less often, e.g., deformation theory and intersection theory.

We shall now see some properties of noetherian / artinian rings. The next proposition should validate the previous remark:

Proposition 2. *Assume that R is an artinian domain. Then, R is a field.*

Proof. We need to show that any $r \in R \setminus 0$ is invertible. Consider the descending chain of ideals $\cdots \subset (r)^{n+1} \subset (r)^n \subset \cdots \subset (r)$. By artinianity, this must stabilize, in particular, there is $n > 1$ such that $(r)^n = (r)^{n-1}$. This means that $r^{n-1} = r^n \cdot u$ for some $u \in R$. Again, this means that $r^{n-1}(ru - 1) = 0$. Since R is a domain, we conclude that $(ru - 1) = 0$, because $r \neq 0$. So u is the inverse of r. □

On the contrary, there are many noetherian domains which are not fields (see later).

Proposition 3. *Assume that M is noetherian (resp. artinian) R-module. Then, any submodule $N \subset M$ is noetherian (resp. artinian) and every quotient module $M \twoheadrightarrow N$ is noetherian (resp. artinian).*

Proof. We show this for noetherian only, the proof being the same for artinian (just reverse the inclusions). If $N \subset M$ is a submodule, and if

$$N_1 \subset N_2 \subset \cdots$$

is an ascending chain in N, then it is also in particular an ascending chain in M. Since M is noetherian, we know that this stabilizes.

Similarly, consider a surjection $\pi\colon M \twoheadrightarrow N$ and let

$$N_1 \subset N_2 \subset \cdots$$

be an ascending chain on N. Then, $\pi^{=1}(N_i)$ defines an ascending chain on M, which must then stabilize. So it follows that also $N_1 \subset N_2 \subset$ stabilizes (due to the surjectivity of π). $\square$

Proposition 4. *Consider a short exact sequence of R-modules*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

If A and B are noetherian (resp. artinian) then C is noetherian (resp. artinian).

Note that if B is noetherian, then both A and C are noetherian by the previous result (the same for artinian).

Proof. Pick any ascending chain $B_1 \subset B_2 \subset \cdots B_n \subset \cdots \subset B$ of submodules of B. Then, $f(B_i)$ is an ascending chain of submodules of C. In particular, it has to stabilize, becasue C is noetherian. Similarly, $B_i \cap A$ yields an ascending chain of submodules of A, which must therefore stabilize. So, there is $N > 0$ such that for every $i \geq j \geq N$ we have $f(B_i) = f(B_j)$ and $B_i \cap A = B_j \cap A$. The result now follows from the following lemma: $\square$

Lemma 5. *Consider a short exact sequence of R-modules*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Let $B_1 \subset B_2 \subset B$ be submodules such that $f(B_1) = f(B_2)$ and $B_1 \cap A = B_2 \cap A$. Then $B_1 = B_2$.

Proof. Assume $B_1 \neq B_2$ and pick $b_2 \in B_2 \setminus B_1$. Now, $f(b_2) \in f(B_2) = f(B_1)$ so there is $b_1 \in B_1$ such that $f(b_1) = f(b_2)$, which means that $b_2 - b_1 \in \ker(f) = A$. Since $b_1 \in B_2$ because $B_1 \subset B_2$ then $b_2 - b_1 \in A \cap B_2 = A \cap B_1$, which implies that $b_2 - b_1 \in B_1$, hence $b_2 \in B_1$, which is a contraddiction. $\square$

Proposition 6. *Assume that R is noetherian ring (resp. artinian) and let M be a finitely generated R-module. Then, M is noetherian (resp. artinian).*

Before proving this, we check by induction the following lemma:

Lemma 7. *Assume that R is noetherian (resp. artinian). Then, R^n is noetherian (resp. artinian) as R-module.*

Proof. By induction on n. If $n = 1$ there is nothing to prove. Now, we have a short exact sequence

$$0 \to R^n \to R^{n+1} \to R \to 0,$$

and we use the induction hypothesis plus the previous result to conlcude. $\square$

Corollary 8. *Let R be noetherian (resp. artinian). Then any finitely generated R-module is noetherian (resp. artinian).*

Proof. In fact, a module is finitely genereated if and only if it admits a surjection $R^n \twoheadrightarrow M$. Then we can apply the previous results to conclude. $\square$

In particular, note the following: if R is an artinian ring, then every finitely generated module over it is noetherian (by the fact that R is noetherian and the previous proposition).

We now characterize Noetherian modules in a very convenient way

Proposition 9. *Let M be a R-module. Then, M is Noetherian if and only if every submodule of M is finitely generated.*

Proof. We show the first implication. So we assume that M is noetherian and we show that all its submodules are finitely generated. Since any submodule $N \subset M$ is noetherian, we can just show that if M is noetherian, then it is finitely genereated. So, suppose that M is not finitely generated. This means that for every $m_1, m_2, \cdots m_k \in M$ we have $\sum_i Rm_i \neq M$. Now, we construct an infinite ascending chain as follows: pick $m_1 \in M \setminus 0$ and put $M_1 = Rm_1$. By assumption, there is $m_2 \in M \setminus M_1$, and we put $M_2 = Rm_1 + Rm_2$. Note that $M_1 \subset M_2$ is a strict inclusion. Then again, by assumption, $M \setminus M_2$ is not empty, and we can pick $m_3 \in M \setminus M_2$ and reiterate the process.

Now we prove the other implication. So, we assume that all the submodules of M are finitely generated. Pick an ascending chain $M_1 \subset M_2 \subset M_3 \cdots \subset M$ and put $M' = \bigcup_i M_i \subset M$. Then M' is a submodule of M, therefore, by assumption, it is finitely generated, say by $m_1, \cdots, m_k \in M'$. But then, there must be a $n > 0$ such that $\{m_1, \cdots, m_k\} \subset M_n$. Therefore, $M_n = \bigcup_i M_i \subset M$ which implies that the chain stabilizes. $\square$

We now prove Hilbert's basis theorem:

Theorem 10. *If R is a noetherian ring, then $R[x]$ is a noetherian ring as well.*

Proof. We need to show that every ideal $I \subset R[x]$ is finitely generated. For any $f \in R[x]$ let $c(f)x^n$ be its leading coefficient, that is, $\deg(f) = n$ and $f - c(f)x^n$ has degree $< n$. We consider the ideal $J \subset R$ generated by $\{c(f) \colon f \in I\}$. As R is Noetherian, we deduce that J is finitely generated. In particular, there are finitely many $f_1, \cdots, f_r \in I$ such that $c(f_1), \cdots, c(f_r)$ generate J. Now, up to multiplying by some x^k each of the f_i, we can assume that $\deg(f_i) = \deg(f_j) = n$ for every i, j.

Consider now the subgroup $I^{\leq n} \subset I$ given by

$$I^{\leq n} := \{f \in I \colon \deg(f) \leq n\}.$$

This is easily seen to be a R-module. Moreover

$$I^{\leq n} \subset R[x]^{\leq n} = \{f \in R[x] \colon \deg(f) \leq n\} \cong R^{n+1}.$$

Since R is noetherian, R^{n+1} is noetherian too, and therefore also $I^{\leq n}$ is a noetherian R-module. As such, it is generated over R by finitely many elements $g_1, g_2, \cdots, g_s$. We now prove that

$$f_1, f_2, \cdots, f_r, g_1, g_2, \cdots, g_s$$

generate I as a $R[x]$ module. Pick $f \in I$. If $\deg(f) \leq n$ then $f \in I^{\leq n}$ is a R-linear combination of $g_1, g_2, \cdots, g_s$. Now we do induction on $\deg(f) > n$. Recall that $\deg(f_i) = n$ for every i. Since $c(f) \in J$ we can write

$$c(f) = a_1 c(f_1) + a_2 c(f_2) + \cdots a_r c(f_r).$$

It follows from this that

$$f - x^{\deg(f)-n}(a_1 f_1 + a_2 f_2 + \cdots + a_r f_r)$$

has degree strictly smaller than $\deg(f)$, so we can apply the inductive hypothesis. $\square$

Corollary 11. *For any field K and any ideal $I \subset K[x_1, x_2, \cdots, x_n]$ the quotient ring $K[x_1, \cdots, x_n]/I$ is noetherian.*

The rings of the form $K[x_1, \cdots, x_n]/I$ correspond to the function rings of affine varieties, which are the building block of algebraic geometry. So, basically, most of algebraic geometry is done under the noetherian assumption.

Definition 12. We say that a module M is simple if its only submodules are 0 and M. We say that a submodule $N \subset M$ is maximal if M/N is simple. We say that $N \subset M$ is minimal if N is simple.

For a module M, a composition series is a chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M \tag{0.1}$$

such that M_i/M_{i+1} is a non-zero simple module for every $0 \leq i \leq \ell - 1$. It is easy to see that M has a composition series if and only if it is both Artinian and Noetherian. We call ℓ the length of the composition series. We shall prove:

Theorem 13 (Jordan-Hölder)**.** *Suppose that M has a composition series like* (0.1)*. Let*

$$0 = M'_0 \subset M'_1 \subset M'_2 \subset \cdots \subset M'_s = M$$

be another composition series for M. Then, $s = \ell$ and the sequences of quotients

$$M_1, M_2/M_1, \cdots, M_\ell/M_{\ell-1}$$

and

$$M'_1, M'_2/M'_1, \cdots, M'_\ell/M'_{\ell-1}$$

coincide up to reordering.

Proof. We prove the result by induction on ℓ. If $\ell = 1$ then M is a simple module, so there is nothing to prove. So we assume that we know the result up to $\ell - 1$ and we want to show that it is true also for ℓ. So we assume that M has a composition series like in (0.1) of length ℓ. We differentiate between three cases.

Case one: there are $0 < i < \ell$ and $0 < j < s$ such that $M_i = M'_j$. In this case, we can apply the inductive hyphotesis to see that $i = j$ necessarily. Moreover, the quotients

$$M_1, M_2/M_1, \cdots, M_i/M_{i-1}$$

and

$$M'_1, M'_2/M'_1, \cdots, M'_i/M'_{i-1}$$

must coincide up to reordering, thanks again to the inductive hypothesis. Finally, we note that

$$0 \subset M_{i+1}/M_i \subset M_{i+2}/M_{i+1} \subset \cdots M/M_{\ell-1}$$

and

$$0 \subset M'_{i+1}/M'_i \subset M'_{i+2}/M'_{i+1} \subset \cdots M/M'_{\ell-1}$$

are both composition series for the same module $M/M_i = M/M'_i$ of length strictly smaller than ℓ. Again by the induction hypthotesis, we deduce that $\ell = s$ and that the sequence of quotients coincides. But by the third isomorphism theorem, these sequences are nothing but

$$M_{i+1}/M_i, \cdots, M/M_{\ell-1}$$

and

$$M'_{i+1}/M'_i, \cdots, M/M'_{\ell-1}$$

so we are done.

Case two: there is $0 < i < \ell$ and $0 < j < s$ such that $M_i \subset M'_j$. Note that Case two is more general than case one. We shall connect it to case one, as follows. Note that the hypothesis of case 2 imply that $M_1 \subset M'_{s-1}$. We shall now construct another composition series for M: first, we find a composition series for M'_{s-1}/M_1. Why we can (by induction). This yields a composition series

$$0 \subset M_1 \subset M''_2 \subset M''_3 \subset \cdots M''_t \subset M'_{s-1}.$$

Again by induction, we now that $t = s - 2$ and that the sequence of quotients

$$M_1, M''_2/M_1, M''_3/M''_2, \cdots, M'_{s-1}/M''_{s-2}$$

is the same (up to reordiring) of

$$M'_1, M'_2/M'_1, M'_3/M'_2, \cdots, M'_{s-1}/M'_{s-2}.$$

Now, we consider the two composition series (0.1) and the new one just constructed

$$0 \subset M_1 \subset M''_2 \subset M''_3 \subset \cdots M''_{s-2} \subset M'_{s-1} \subset M_s = M.$$

Since they agree at $i = 1$, we can apply case one to deduce that $s = \ell$ and that all the quotients coincide.

Case three: This is the last case, which happens when M_1 is not contained in M'_{s-1}. We shall also link this to case one. The fact that M_1 is simple implies that $M_1 \cap M'_{s-1} = 0$ and the fact that M_{s-1} is maximal implies that $M'_{s-1} + M_1 = M$. Therefore, $M \cong M_1 \oplus M'_{s-1}$. We consider the following composition series for M:

$$0 \subset M_1 \oplus 0 \subset M_1 \oplus M'_1 \subset \cdots \subset M_1 \oplus M'_{s-2} \subset M$$

whose quotients are

$$M_1, M'_1, M'_2/M'_1, M'_3/M'_4, \cdots, M'_{s-1}/M'_{s-2}.$$

Since the first term of this coincides with the first term of (0.1), we can apply case one and conclude that $\ell = s$ all the quotients coincide. Since $M_1 = M/M'_{s-1}$ we conclude the proof. $\square$