EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 9	22 November 2024

The exercise marked by ♠ is this week's bonus exercise. You can hand in your LaTeX-solutions on Moodle until Wednesday the 4th of December at 6pm sharp.

Exercise 1. Show the following:

(1) Prove that the only prime ideal of height zero in a domain is the ideal (0).
(2) Prove that a prime ideal of height 1 in a UFD is principal.
(3) Compute the prime ideals of height zero in $\mathbb{R}[x,y]/(xy)$.
[*Hint*: Recall that there is a one-to-one correspondence between the prime ideals R containing I and the prime ideals of R/I.]

Exercise 2. Show the following:

(1) If R is a domain with $\dim R = 0$, then R is a field.
(2) We say that a ring R is reduced if there are no nilpotent elements in R. That is, if $r \in R$ is such that $r^n = 0$ for some n, then $r = 0$. Give an example of a reduced ring R of dimension zero which is not a field.

Exercise 3. Solve the following problems:

(1) Prove that every Artinian ring has dimension 0.
(2) Compute the dimension of the ring $\mathbb{Z}[x]/(4, x^2)$.

Exercise 4. Let R be a PID which is not a field. The goal of this exercise is to show that $\dim R[x] = 2$ (in particular $\dim \mathbb{Z}[t] = \dim k[x,y] = 2$).

- Show that $\dim R[x] \geq 2$.
- Let $\mathfrak{p}$ be a non-zero prime ideal of $R[x]$. Show that $\mathfrak{p}$ has height 1 if and only if it is principal.
- Let $K = \mathrm{Frac}(R)$. For any prime ideal $\mathfrak{p}$ in $R[x]$, define $\mathfrak{p}^e$ to be the ideal of $K[x]$ generated by the elements of $\mathfrak{p}$. Show that if $\mathfrak{p}$ is a prime ideal of height 2, then $\mathfrak{p}^e = K[x]$. Conclude that there exists $\pi \in R$ irreducible such that $\pi \in \mathfrak{p}$.
[*Hint: Recall the notion of primitive polynomial, and the statements around Gauss' lemma (see for example proposition 3.8.13 in the "Anneaux et corps" notes).*]
- Conclude that any prime ideal of height 2 is maximal, and deduce that $\dim(R[x]) = 2$.

[*Remark:* It is a general fact that given a Noetherian commutative ring R of finite Krull dimension, $\dim(R[x]) = \dim(R) + 1$. This is not so complicated once we have proven Krull's Hauptidealsatz, but we unfortunately do not have the time to cover this in the course. See the course "Modern algebraic geometry" (or any book in commutative algebra) if you want to know more about this.]

Exercise 5 (Nakayama's Lemma)**.** Let R be a commutative ring and let M be a finitely generated R-module. Show the following:

(1) Let I be an ideal of R such that $IM = M$. Then there exists $x \in 1 + I$ such that $xM = 0$.
[*Hint:* The proof is similar to the direction $(3) \Rightarrow (1)$ in Proposition 6.2.3 of the lecture notes.]

(2) Suppose now that the ring R is local, i.e., that there is a unique maximal ideal $\mathfrak{m}$ of R. Show that if $\mathfrak{m}M = M$, then $M = 0$.
(3) For a ring R denote by $\mathrm{Jac}(R)$ the intersection of all maximal ideals of R; this is called the *Jacobson radical* of R (note also that $\mathrm{nil}(R) \subseteq \mathrm{Jac}(R)$). Show that if there is an ideal $I \subset \mathrm{Jac}(R)$ such that $IM = M$, then this implies that $M = 0$. This generalizes the previous point to any ring.

[*Hint:* Prove that in (2), (3) the element x, whose existence is assured by (1), is in fact invertible.]

Remark 0.1. Nakayama's lemma is a **EXTREMELY** powerful tool in commutative algebra and algebraic geometry, so keep it mind this exists. You should really (really) remember it!

To give a hint of its tremendous power, recall you had an exercise about showing that if R is a commutative ring, M a finitely generated module and $f\colon M \to M$ a surjective endomorphism, then f is an isomorphism. Actually, the statement follows immediately by considering M as an $R[x]$-module via $x \cdot m = f(m)$, and taking $I = (x)$ in (1).

Recall that when you proved it in an early exercise sheet, you had a Noetherian assumption on R (and it was fundamental to the proof, have fun trying to prove it directly without this assumption!). With this argument, you don't need it!

Exercise 6. Let R be a commutative ring which is an integral domain but not a field, and let F be the fraction field of R. Show that F is not finitely generated as an R-module.

Exercise 7. Let $R = \mathbb{F}_q[[t]]$ be the ring of power-series in the variable t over the finite field with q elements $\mathbb{F}_q$.

Recall that as a set, R is the set of formal power-series $f = \sum_{n\geq 0} a_n t^n$ with coefficients $a_n \in \mathbb{F}_q$. For two such power series, $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} b_n t^n$, one defines the addition to be the power-series $\sum_{n\geq 0}(a_n + b_n)t^n$ and multiplication to be the power-series $\sum_{n\geq 0}(\sum_{k=0}^{n} a_k b_{n-k})t^n$. Recall (or do) the two following exercises from "Anneaux et corps":

(1) If $f \in R \setminus (t)$, then f is invertible (and hence R is a local ring with maximal ideal (t)).
(2) A formal Laurent series over the field $\mathbb{F}_q$ is defined in a similar way to a formal power series, except that we also allow finitely many terms of negative degree. That is, series of the form $f = \sum_{n\geq N} a_n t^n$ where for some $N \in \mathbb{Z}$. Define a natural ring structure on this set and show that with this ring structure the ring of formal Laurent series over $\mathbb{F}_q$, usually denoted $\mathbb{F}_q((t))$, is equal to the fraction field of R.

Now let us go to the actual exercise:

(3) Show that $\mathrm{trdeg}_{\mathbb{F}_q}(\mathrm{Frac}(R))$ is infinite.
[*Hint*: show that $\mathbb{F}_q(t_1, \ldots, t_r)$ is countable, and R is not.]
(4) Show that $\dim R = 1$ and hence show that Theorem 6.1.12 in the course notes does not work with non-finitely-generated algebras.

Exercise 8. ♠ Let R be a Noetherian local ring (i.e. it has a unique maximal ideal) with maximal ideal $\mathfrak{m}$, and set $k = R/\mathfrak{m}$. Furthermore, fix a finitely generated module M over R.

(1) Show that if $f\colon M \to N$ is a morphism of finitely generated modules, such that the induced map $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective. Show that f is then surjective.
(2) A *minimal free resolution* of M is a resolution

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \cdots \to F_0$$

of M such that for all n, F_n is free of finite rank and $\mathrm{im}(d_n) \subseteq \mathfrak{m}F_{n-1}$. Show that M admits a minimal free resolution.

(3) Fix a minimal free resolution $F_\bullet$ of M. Then show that for all $n \geq 0$, $\mathrm{Ext}^n(M, k) \neq 0$ if and only if $F_n \neq 0$.

(4) Deduce the surprising fact that if $\mathrm{Ext}^{n+1}(M, k) \neq 0$, then $\mathrm{Ext}^n(M, k) \neq 0$.

(5) Show that a finitely generated projective module over R is free.

Hint for this exercise (and for your potential future life as a commutative algebraist/algebraic geometer): Nakayama's lemma is your friend.